The demand for liquid asset with uncertain lumpy expenditures*

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Abstract

We consider an inventory model for liquid asset where the per period net expenditures have two components: one that is frequent and small and another that is infrequent and large. We give a theoretical characterization of the optimal management of liquid asset as well as the implied observable statistics. We use our characterization to understand aspects of the management of currency of Austrian households, as well as the management of liquid asset of Italian investors.

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1 Introduction

This paper studies some implications of lumpy uncertain purchases for the management of liquid asset in the context of inventory theoretical models. By lumpy purchases we mean large-sized expenditures that must be paid with a liquid asset. The paper accomplishes three objectives. First, it breaks some new ground on the mathematical analysis of inventory models. The solution of the model with jumps turns out to be non-trivial. Second, it shows that some of the theoretical predictions are in stark contrast with those of canonical inventory models. In particular, a basic feature introduced by the presence of lumpy purchases is the possibility that liquidity gets withdrawn and spent immediately. Compared to canonical models this feature changes the relationship between the size of liquidity withdrawals and the average liquidity holdings. Equivalently, this impacts the relationship between the average cash holdings and its “scale variable”, e.g. the average expenditures in a period. Third, it brings new evidence to bear on the model predictions concerning households liquidity management using two novel datasets of Austrian and Italian households.\footnote{The data for Austria contain detailed information on currency management practices, as well as on the pattern of expenditures paid with currency, in particular their size distribution, the size and frequency of withdrawals and the precautionary holdings of cash. We also look at the management of liquid asset by Italian investors, for a definition of liquid asset that is broader than currency.} We argue that the model is useful to interpret novel evidence on household liquidity management which existing models cannot explain. In this paper we concentrate the discussion on households’ liquidity management mostly because of the data availability, we think that the same ideas have clear implications for the demand of liquid assets by firms.

The standard inventory model solves the problem of an agent trading off the holding cost of an inventory with the cost of adjusting this inventory. In simple models inventories are assumed to be needed to face an exogenous process for consumption of households or net sales of firms. These models are typically set up in continuous time and assume that the need for inventories is described by a process with continuous paths. Examples of these models applied to liquid asset holdings are the seminal work of \textit{Tobin} (1956), \textit{Baumol} (1952), \textit{Miller}
and Orr (1966), and the literature that follows them. In this context, the need for inventories (liquid asset, or currency) is derived from an assumed process for the agents net consumption paid (or received) in cash. When applied to household (as in Tobin (1956), Baumol (1952)) this process is the consumption that they need to pay in cash, and when applied to firms (say as in Miller and Orr (1966)) it is the net revenue of the firms in cash. In this case one of the inventory cost is given by the low return of cash or liquid asset in general. This paper aims to capture the implications for cash-management of having lumpy uncertain purchases: thus we explore the consequences for these type of models of departing from the assumption that net cash consumption has continuous paths, and instead we allow a jump process for the unregulated inventories.

The ideas in this paper can be sketched in the context of a simple model. Suppose that in each period the agent has to finance a consumption in cash of \( c \) per unit of time and also that with a probability per unit of time \( \kappa \) in each period that an agent to make a payment in cash of size \( z \). In this case, the expected consumption to be finance with cash is \( e = c + \kappa z \) per unit of time. One strategy for the agent is to withdraw enough money so that, at least if this happens close after to the time of the withdrawal, these payment can be financed with the cash at hand. This strategy has the advantage of saving on the adjustment cost, but it has the disadvantage of incurring a holding cost of the inventory of cash. The inventory cost increases as the purchase size \( z \) become larger, since the agent has to withdraw more money. The alternative strategy is to withdraw money when the purchase \( z \) happens. This strategy saves in holding cost, since the agent spends the money right away, but it does incur more frequent adjustment cost. This strategy is preferred when the probability \( \kappa \) is small. Thus, as the size of the large purchases \( z \) increases and as they become infrequent (\( \kappa \) small) the optimal policy is to withdraw every time that the large purchase happen. In this simple extreme case the expected value of large purchases \( \kappa z \) has no effect on the average cash holdings. The size of the withdrawal, when triggers by a large purchase, increases by the amount of this purchase, and the amount \( z \) of cash is spent immediately. This, in a sense turns the logic of
the classical model up-side down: cash is not spent slowly and adjustment triggered as they reach the boundary of the $sS$ bands, but instead the random purchase triggers an adjustment and a simultaneous withdrawal and large use of cash. While $\kappa z$ has no effect on the average cash holdings, it has a the full effect in the average size of the withdrawal, and hence $W/M$ is increasing in $\kappa z$. The high value of $W/M$ also implies that the number of withdrawals per unit of time $n$ is small relative to the benchmark of the Baumol-Tobin model for an agent financing the same consumption $e$: this is a natural result, the withdrawals that are triggered by large purchases account for a large share of cash expenditures $e$ and contribute nothing to average money holdings. Additionally, if every random large expenditures triggers a withdrawal, agents on average hold cash at the time of withdrawals, a behavior that can be described as “precautionary”.

There is a large literature in inventory models applied to liquid assets. Most of the literature assumes that the cumulated net cash consumption as continuous path, so that it does not have lumpy purchases or sales. Examples are Baumol (1952), Tobin (1956), Miller and Orr (1966) among many others.\(^2\) One exception is the the work by Bar-Ilan, Perry, and Stadje (2004). The set up of that paper includes lumpy purchases and sales in a more general way that the one in this paper. Yet they do not explore the nature of the optimal decision rules, instead they compute the value for some policies. Instead, in this paper we gave a characterization of the optimal policy, which from the technical point is view is not a trivial matter since it has to address two issues. The form of the inaction set, i.e. whether it is a single interval or the union of disjoint ones- and whether the necessary boundary conditions are also sufficient -which in this case they are not. Additionally, the focus of our paper is different, we concentrate on the implications for the cash management statistics and on the differences with standard models.

We use two surveys from currency management, one from Italian households and one from

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Austrian. Both surveys contain information on the patterns of cash management: the average consumption paid with cash per period $e$, the average cash holdings $M$, average withdrawal size $W$, average number of withdrawals per unit of time $n$, and average cash holdings at the time of withdrawal $M$. Besides that, the Austrian data set contains information on the patterns of purchase size, coming from a consumption diary for the same individuals for which the survey was administered, as described in Mooslechner, Stix, and Wagner (2006). This diary shows that for a non-negligible fraction of individuals, the assumption that large purchases are paid in cash is realistic. We use the diary and survey information to investigate some of the predictions of our model by comparing individuals that differ in the importance of the lumpy component of their expenditures paid with currency. In Section 4.1 we present evidence on several statistics, such as the frequency and size of withdrawal relative to the average holding of currency holdings, that is supportive of the mechanism highlighted in the model.

While we focus on the interpretation where currency has to be used to pay for transactions, the model can be applied also to more aggregate notions of liquid assets. In Alvarez, Guiso, and Lippi (2011) we analyze a class of models where households must use liquid asset to pay for all their expenditures and they face information and/or transaction cost to transfer money from high yield assets to low yield liquid asset. Additionally in Alvarez, Guiso, and Lippi (2011) we consider versions with either non-durable consumption or durable consumptions. In the model where all the expenditures are in non-durable goods – a version of Duffie and Sun (1990) or Abel, Eberly, and Panageas (2007) – so that the expenditures occur at a constant rate between adjustment of liquid assets, implying an average holding of liquid asset similar in several respect to the one in Tobin (1956)-Baumol (1952). In the model where all the expenditures are in durable goods –a variation of Grossman and Laroque (1990)– so that the expenditures are lumpy and occur infrequently, implying that they can be paid without holding any liquid asset between adjustments, and hence average holding of liquid assets is zero. A more realistic model will have expenditures both in non-durable goods –so
that they are continuous— as well as on durable goods -which with either transaction cost of indivisibilities, becomes lumpy and infrequent. The model in this paper analyzes such a set up. In Alvarez, Guiso, and Lippi (2011) we use a panel data of administrative data from the accounts of italian investors during 35 months, as well as cross sections from a household survey and report several patterns consistent with the management of liquid asset of the model presented in this paper. We review this evidence in Section 4.2

Even though the model of this paper is constructed with the application of households cash management in mind, we think that the same ideas apply for model of liquidity management for firms. The idea applied to such a context, is that if firms have large expenditures which have to be paid with liquid asset, they will not affect the average holding of this asset, by the same reasons that in the model for households. We think that this is also a potentially testable implication, by examining panel data of liquid asset holding. From the existing literature we find some confirmation of our hypothesis. Bates, Kahle, and Stulz (2009) run several specifications of panel regressions to explain the ratio of liquid asset to total asset for of U.S. manufacturing firms from 1980 to 2006. After controlling for other determinants of liquid asset holding, they find a negative coefficient on the ratio of acquisitions to assets, which we interpret to be measuring large an unfrequent disbursements of liquid assets.

The working hypothesis of our model is that, in the face of a large purchase that has to be made in cash, the household can obtain the cash after paying the fixed adjustment cost. This is the typical assumption in inventory theoretical models. But consider the other extreme case which gives rise to different cash management: assume that when a large purchase needs to be made the agent is not able to obtain the cash. In this case she will be forced to always carry an extra amount of cash. This is a plausible hypothesis, and indeed it has been used to explain the demand for liquid assets, even for households that simultaneously hold short term debt in Telyukova (2009). One way to distinguish between these hypothesis,

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3Hence while the model generalizes the models considered in Alvarez, Guiso, and Lippi (2011) by having both type of expenditures, it simplifies the set up in two dimensions: the process for expenditures are exogenous and the adjustment cost does not include observation costs.
besides that of looking for their implications in other variables, is to measure directly the
time series properties for deposits, withdrawals and cash expenditures. One data set where
this information is available is the monthly longitudinal panel of about 600 households in
Thailand created by and maintained by Robert Townsend. Preliminary analysis of this data
in Alvarez, Pwasutipaisit, and Townsend (2011) supports the hypothesis of simultaneous
large expenditures and withdrawals. Additionally in Alvarez, Guiso, and Lippi (2011) we
find similar patterns around house purchases using a panel of bank accounts for 40,000
Italian investors.

2 A deterministic model of lumpy purchases

We develop here a simple version of Baumol-Tobin model where the consumption paid in
cash has two deterministic components, one continuous at the rate $c$ per unit of time and
the other discontinuous, with jumps of size $z$, exactly every $1/\kappa$ periods of time. Thus, total
consumption per unit of time is $e \equiv c + z\kappa$, the sum of the cumulative consumption at the
rate $c$ plus the $\kappa$ jumps in consumption, each of them of size $z$. These jumps on the cash
consumption in the model are meant to be a simple representation of the fact that households
purchases varies in size.

The objective function is as in Baumol-Tobin: to minimize the cost $V$ given by $V = RM + bn$
where $M$ are average cash balances, $R$ is the opportunity cost of the cash balances, $n$
is the number of withdrawals per unit of time, and $b$ is the fixed cost paid for each withdrawal.

It turns out that the optimal policy is of one of three types, depending on parameters.
When $\kappa$ is small, the agent withdraws every $1/(i \kappa)$ units of time, where $i \geq 1$. In this case,
there are $i$ withdrawals between jumps in cumulative consumption, and $n > \kappa$. One of the
withdrawals will happen just before the jump $z$, and hence financing the discontinuous part
of consumption is done at no cost. If $\kappa$ is large, the agent makes a withdrawal every $j/\kappa$
units of time, where $j \geq 1$. In this case there are $j$ jumps in cumulated consumption between
successive withdrawals, or \( n < \kappa \). Thus the agent will only "save" on the opportunity cost of the associated consumption \( z \) once every \( j \) jumps between withdrawals. For intermediate values of \( \kappa \), withdrawals happen exactly every \( 1/\kappa \) periods of time, or \( n = \kappa \).

We define two thresholds \( \kappa^* \) and \( \kappa^{**}(z) \), which determines the pattern of cash management:

\[
\kappa^* \equiv \sqrt{\frac{R c}{2b}} \leq \kappa^{**}(z) \equiv \frac{R z + \sqrt{(R z)^2 + 8b Rc}}{4b}.
\]

Note that \( \kappa^* = \kappa^{**}(0) \) and that \( \kappa^{**} \) is strictly increasing in \( z \).

To simplify the description we will assume that certain combination of parameters takes on integer values. Define \( u \) as follows:

\[
u \equiv \max \left\{ \sqrt{\frac{R c}{2b}}, \sqrt{\frac{\kappa^2 2b}{R (c + \kappa z)}} \right\}.
\]

For the description of the optimal policy we let \( W \) be the average withdrawal size, so \( W/M \) is the ratio of average withdrawal to average stock of money.

**Proposition 1.** Assume that if \( u > 1 \), then \( u \) is an integer. Then the optimal decision rules and value of the objective function \( C \) are given by:

If \( \kappa < \kappa^* \) : \( n = \sqrt{\frac{R c}{2b}} > \kappa \), \( \frac{W}{M} = \frac{c + \kappa z}{c} \), and \( V = \sqrt{2 R b c} \),

If \( \kappa^* \leq \kappa \leq \kappa^{**}(z) \) : \( n = \kappa \), \( \frac{W}{M} = \frac{c + \kappa z}{c} \) and \( V = \frac{R c}{2} + b \kappa \),

If \( \kappa > \kappa^{**}(z) \) : \( n = \sqrt{\frac{R (c + z\kappa)}{2b}} < \kappa \), \( \frac{W}{M} = \frac{c + \kappa z}{c + z(\kappa - n)} \) and

\[
V = \sqrt{2 R b (c + \kappa z)} - \frac{R z}{2}.
\]

We note that using the accounting identity \( W n = c + \kappa z \) and the information in the proposition one can find the values of \( W \) and \( M \) separately. The interpretation of this proposition is as follows: when \( \kappa \) is small relative to what determines the frequency of
withdrawals in Baumol-Tobin, then the jumps can be made coincide to be just after with one of the many withdrawals. In this case, the agent will withdraw the extra amount \( z \) and spend it immediately. Thus, the part of consumption associated with the jump does not incur any opportunity cost \( R \), since the case is held by an instant. Also, the agent does not incur any extra fixed cost \( b \), since it has to withdraw to finance the continuous expenditure anyway. As a consequence, the decision for the agent on the number of withdrawals \( n \) is exactly as in Baumol-Tobin, but as if the consumption to be financed is \( c \), instead of \( c + \kappa z \). This can be seen for the expression for \( n \) and \( V \). The expression for \( W/M \) is larger than 2, since at the of a jump in consumption the agent withdraws an extra amount \( z \), but this is spent immediately, and hence does not contribute to the average money balances \( M \). On the other hand, consider the case where \( \kappa \) is large, so that the agent will like to withdraw several times between jumps. In this case, only the first of the jump, the one that occurs immediately after a withdrawal will have no opportunity cost associated with it. Otherwise, the decisions are as if the agent has to finance \( c + \kappa z \) in the Boumol-Tobin model. This can be seen in the expression for \( n \), which is the same as in Baumol-Tobin, and in the one for \( V \), which is identical, except that it subtracts the "savings" in the opportunity cost for one jump per period. The ratio of \( W/M \) depends on how large this jumps are, i.e. on \( z \). The following extreme case may help to understand the model for large \( \kappa \). Assume that \( \kappa \) is very large, but \( z \) is very small, so there are very frequent jumps of small size, keeping the product \( z\kappa = \gamma \) positive and finite. As the jumps become very small, the model is identical to Baumol-Tobin, with total consumption \( c + \kappa z \). This can be seen in the case where \( \kappa > \kappa^*(z) \), and taken the limit to \( z \) to zero. We now develop the comparison with Baumol-Tobin in detail.

Let’s denote by \( n_{BT} = \sqrt{\frac{R (c + z\kappa)}{2b}} \), the optimal decision if one were to measure the total cash consumption \( c + z\kappa \) and assume that it is continuous as in Baumol-Tobin. Also we recall that in Baumol-Tobin, since cash consumption is constant per unit of time (\( z = 0 \)), then the ratio of withdrawals to money stock is 2. We then compute the ratio \( n/n_{BT} \) and
(W/M)/2 as a function of κ which varies as follows:

If \( \kappa \leq \kappa^* \) \( \Rightarrow \) \( \frac{n}{n_{BT}} = \sqrt{\frac{c}{c + \kappa z}} \), \( \frac{W/M}{2} = \frac{c + \kappa z}{c} \),

If \( \kappa^* < \kappa < \kappa^{**}(z) \) \( \Rightarrow \) \( \frac{n}{n_{BT}} = \sqrt{\frac{\kappa^2 2 b}{R (c + \kappa z)}} \leq 1 \), \( \frac{W/M}{2} = \frac{c + \kappa z}{c} \),

If \( \kappa \geq \kappa^{**}(z) \) \( \Rightarrow \) \( \frac{n}{n_{BT}} = 1 \), \( 1 \leq \frac{W/M}{2} = \frac{c + \kappa z}{c + \kappa z - z \sqrt{\frac{R (c + \kappa z)}{2 b}}} < \frac{c + \kappa z}{c} \).

We note that, in terms of the statistics W/M and \( n/n_{BT} \) the implications of the model when \( \kappa < \kappa^* \) depend only the value of \( \gamma \), and not separately on \( \kappa \) and \( z \), a feature that will be shared by the model in Section 3.3. There are two extreme cases that we find useful to highlight. We keep the product \( z \kappa = \gamma > 0 \), strictly positive and finite. Then we have:

\[
\lim z = 0, \lim \kappa = \infty \quad \Rightarrow \quad \frac{n}{n_{BT}} = 1, \quad \frac{W/M}{2} = 1,
\]

\[
\lim z = \infty, \lim \kappa = 0 \quad \Rightarrow \quad \frac{n}{n_{BT}} = \sqrt{\frac{c}{c + \gamma}}, \quad \frac{W/M}{2} = \frac{c + \gamma}{c}.
\]

3 A stochastic model

We consider a model where consumption has three components: one is deterministic at a constant rate \( c \) per unit of time, as in our previous model. The second component represent large purchases: we assume that the jump process occurs with probability \( \kappa \) per unit of time, and that when it happens cumulated consumption increases by an amount given by the parameter \( z > 0 \). With this parameterization, expected consumption per period, say per year, equals \( e = c + \kappa z \). The third component, which we add mostly as a rhetorical device, are random variation is net cash consumption, with variance \( \sigma^2 \) per unit of time. If we denote cumulative consumption paid in cash by \( C(t) \) we assume that \( dC(t) = c dt + z dN + \sigma dB \), where \( N(t) \) is the poisson counter, and \( B(t) \) is an standard Brownian motion. If we interpret \( dC \) as the consumption during a period of length \( dt \), we note that, when \( \sigma > 0 \), it can
negative. This is to capture, as in the seminal model of cash management of firms by Miller and Orr (1966), income that is received in cash.

We also assume that, with Poisson arrival rate \( p \) per unit of time, the agent has an opportunity to adjust her cash balances without paying the cost \( b \). We have explore this feature in Alvarez and Lippi (2009). We included here because it shared some implications with the model with large purchases. Specifically, we show below, that cash management data alone (such as frequency and size of withdrawals, average cash holdings, etc) cannot identify separately \( p \) from \( \kappa \). On the other hand, having data on the size distribution of cash purchases can help identify these parameters.

The aim of this extension is to explore the implications of an alternative reason for "precautionary" type of behavior. In this model, there are three types of withdrawals, those that occur when \( m \) reaches zero, and those that occur at the time of a jump in consumption if \( m < z \), and those that occur if the agent has a free withdrawal opportunity. The idea is that at times when cumulated consumption jumps (i.e. a large purchase occurs), if the money balances at hand \( m \) are not large enough to pay for the sudden increase in cash consumption, i.e. if \( m < z \), then the agent will withdraw cash, even if cash has not reached zero. Otherwise, the nature of the optimal policy is the same, after withdrawal agents set their cash balances to the optimal replenishment level \( m^* \).

The standard inventory model has unregulated inventory following a process with continuous paths. If we let \( C \) be the cumulated unregulated process, we have that \( C(t) = ct \) for constant \( c > 0 \): Baumol (1952), Tobin (1956), Jovanovic (1982), Alvarez and Lippi (2009); \( C(t) = \sigma B_t \) for constant \( \sigma > 0 \) and \( B_t \) a standard BM in Miller and Orr (1966), Miller and Orr (1968), Eppen and Fama (1969), Weitzman (1968); \( C(t) = ct + \sigma B_t \) for constants \( c, \sigma > 0 \): Constantinides and Richard (1978), Constantinides (1978), Frenkel and Jovanovic (1980), Harrison, Sellke, and Taylor (1983), Harrison and Taskar (1983), Sulem (1986), Bar-Ilan (1990), and \( dC(t) = c(x)dt + \sigma(x)dB \) in Baccarin (2009). Yet there are several exceptions. Milbourne (1983) model is set up discrete time and made no special assumptions about the
process for net cash-holdings. If we let $C$ be the cumulated unregulated process, and we let $N$ be the counter of a Poisson process, we have that $dC(t) = zdN$ from state dependent $F(x, z)$ in Song and Zipkin (1993) (where the state $x$ is a finite Markov chain) and in Archibald and Silver (1978). The paper with the closely related, but more general set up, is Bar-Ilan, Perry, and Stadje (2004), which assumes $dC(t) = \mu dt + \sigma dB + z^u dN^u - z^d dN^d$ and where where $B$ is a standard Brownian motion, $z^i \geq$ are the up and down jumps, and $N^i$ are the counter of two Poisson processes with possible different constant intensities, and where the jump sizes are have general distribution which includes exponential distributions. This paper also has a more general adjustment cost, including fixed and variable cost, that differ from deposits and withdrawals.

Another related literature some paper that -as this paper- model the cash management behavior and -unlike this one- also model the choice of means of payments. An early example is Whitesell (1989), and Bounie and François (2010) a more recent one. In that set up the cost associated with the use of cash and other payments depends on the size of the purchase. Purchases are assumed to come in different size -as in this paper. Yet, to simplify the analysis Whitesell (1989) assumes that “transactions of each size occur at a uniform rate over a unit period”. This assumption allows to study both cash management and the choice of means of payments, clearly an important feature. Yet from an empirical standpoint, this assumption is very much at odds with the fact that large purchases occurs very infrequently, especially relative to the frequency of withdrawals of cash, the relevant comparison for the model. For instance, in Austria the frequency of purchases paid in cash larger than 400 euros is below 2.5 per year, where the number of withdrawals is higher than 20 per year.\(^4\)

We show below that solving the Bellman equation is more involved than in the standard case where the unregulated inventory (cash in this case) follow a process with continuous path. This requires to solve solve a delay-differential equation, as opposed to an ordinary

\(^4\)Additionally, the deterministic feature of the purchases implies that cash is run down to zero before a withdrawal, a feature that runs counter the measurement that at the time of withdrawals Austrians individuals have, in average, an amount of cash that is larger than 20% of their average cash balances.
differential equation. While in Section 3.1 we present an algorithm to solve for the parameters that fully characterize the Bellman equation, we do not have a simple close for solution for the thresholds that describe the optimal policy \( (m^*, m^{**}) \), as we did for the case with no jumps in Alvarez and Lippi (2009). Of course if the jumps were small, i.e. if \( z \) is small, the statistics of interest will not be affected. In particular, we show that in the limit as \( z \rightarrow 0 \) while keeping \( \kappa \times z \) constant, the model reduces to the one with continuous consumption. Thus, in Section 3.3 we will concentrate on the case of large but infrequent jumps, i.e. large \( z \) and small \( \kappa \), which mirrors the deterministic case of \( \kappa < \kappa^* \). We describe the nature of the optimal policy for this case, as well as the implications for several cash management statistics. Finally we present some preliminary evidence on the plausibility of the features emphasized by this model using a data set of Austria households.

### 3.1 Bellman Equation

We consider a trigger policy described by two thresholds: \( m^* \), the value of cash after and adjustment, and \( m^{**} \), the value of cash that triggers a deposit. Non-negativity of cash triggers a withdrawal. After a deposit or a withdrawal, agents return to the value \( m^* \). Thus, the Bellman equation in the interior of the range of inaction, given by \( 0 < m < m^{**} \), becomes:

\[
\begin{align*}
    rV(m) &= Rm + p \left[ \min_{\hat{m}} V(\hat{m}) - V(m) \right] + \frac{\sigma^2}{2} V''(m) \\
    &\quad + \kappa \min \left[ b + \min_{\hat{m}} V(\hat{m}) - V(m) , \, V(m - z) - V(m) \right] \\
    &\quad + V'(m) (-c - \pi m)
\end{align*}
\]

The term \( \min [b + \min_{\hat{m}} V(\hat{m}) - V(m) , \, V(m - z) - V(m)] \) takes into account that after the jump in consumption the agent can decide to withdraw cash, or otherwise her cash balances becomes \( m - z \). We let

\[
m^* \equiv \arg \min_{\hat{m}} V(\hat{m}) , \text{ and } V^* \equiv V(m^*).
\]  

(1)
If the value function is differentiable, we have that

\[ V'(m^*) = 0. \]  \hspace{1cm} (2)

Non-negativity of cash implies that

\[ V(m) = V^* + b \text{ for } m \leq 0. \]  \hspace{1cm} (3)

For the range \( 0 \leq m \leq z \) we look for a solution of the form of an Ordinary Differential Equation (ODE):

\[ (r + p + \kappa) V(m) = Rm + (p + \kappa) V^* + \kappa b + V'(m)(-c - \pi m) + \frac{\sigma^2}{2} V''(m), \]  \hspace{1cm} (4)

since in this range every jump triggers a withdrawal. This feature of the solution is as in Bar-Ilan, Perry, and Stadje (2004), which refer to this adjustments as adjustment triggers by downcrosses. Instead for the range \( z \leq m \leq m^{**} \), we have a Delay-Differential Equation (DDE):

\[ (r + p + \kappa) V(m) = Rm + pV^* + \kappa V(m - z) + V'(m)(-c - \pi m) + \frac{\sigma^2}{2} V''(m), \]  \hspace{1cm} (5)

since in this range after a jump cash balances are positive. If cash reaches the value of \( m^{**} \), then it triggers a deposit of size \( m^{**} - m^* \) after paying the fixed cost \( b \). Thus we have:

\[ V(m^{**}) = V(m^*) + b, \]  \hspace{1cm} (6)
\[ V(m^{**}) = V(m) \text{ for all } m \geq m^{**}. \]

If \( V(\cdot) \) is differentiable at \( m = m^{**} \), then we get that

\[ V'(m^{**}) = 0, \]  \hspace{1cm} (7)
a condition typically referred to as “smooth pasting”. We notice that, in general, it will not be differentiable at this point if \( \sigma = 0 \).

We can further characterize the Bellman equation for \( V \) for given policy described by thresholds \((m^*, m^{**})\) by splitting the range of inaction in intervals of length \( z \). The idea is that, at a given point \( m \), the value function depends on the local evolution around \( m \) and in the value that it will take after a jump, i.e. at \( m - z \). But since cash is non-negative, at when \( m \in [0, z] \), any jump will lead to a withdrawal, and hence, given \( V^* \), the value function only depend on its local evolution, i.e. it is a second order (first if \( \sigma = 0 \)) linear ODE described by equation (4). Then, given the solution of the value function in the lower segment, one can construct the segments corresponding to higher values of \( m \) recursively, which themselves solve a system of ODE’s described by equation (5). In the case of \( \pi = 0 \) the ODEs have constant coefficients. The value matching equation (7)-equation (3)-equation (1) provide three boundary conditions. The continuity of the level, first derivative (and second if \( \sigma > 0 \)) across each segment, provides additional boundary conditions.

**Proposition 2.** Assume \( \pi = 0 \). Given two thresholds \( 0 < m^* < m^{**} \) the value of following such a policy can be described by \( J \) functions \( V_j \):

\[
V(m ; m^*, m^{**}) = V_j(m) \text{ for } m \in [z_j, \min \{ z(j + 1), m^{**} \}],
\]

where

\[
V_j(m) = A_j + D_j(m - z_j) + \sum_{k=1,2} \sum_{i=0}^j B_{j,k}^i e^{\lambda_k(m-z_j)} (m-z_j)^i
\]

where \( \lambda_k \) is the solution of \( r + p + \kappa = -c\lambda + \frac{\sigma^2}{2} \lambda^2 \) for \( k = 1, 2 \) and where the constants \( A_j, D_j, B_{j,k}^i \) for \( j = 0, 1, 2, ..., J-1, i = 1, ..., j, \) and \( k = 1, 2 \) solve a block recursive system of linear equations described in the proof.

Using Proposition 2 we can write the optimality of the return point equation (2) and the
smooth pasting condition equation (7) imply:

\[ 0 = V'(m^*; m^*, m^{**}) = D_{j^*} + \sum_{k=1,2} \sum_{i=0}^{j^*} B_{j^*,i}^k e^{\lambda_k (m^* - z_{j^*})} [\lambda_k + i (m^* - z_{j^*})i - 1], \quad (10) \]

\[ 0 = V'(m^{**}; m^*, m^{**}) = D_{j-1} + \sum_{k=1,2} \sum_{i=0}^{J-1} B_{j-1,i}^k e^{\lambda_k (m^{**} - z(J-1))} [\lambda_k + i (m^{**} - z(J-1))i - 1], \quad (11) \]

where \( j^* \) is the smallest integer such that \( m^* < (j^* + 1)z \), and where equation (11) applies only if \( \sigma > 0 \). Since Proposition 2 shows that the constants \( \{A_j, D_j, B_{j,i}^k\} \) are a function of \( (m^*, m^{**}) \), we can regard equation (10) and equation (11) as a system of two non-linear equation determining \( (m^*, m^{**}) \). In the case of \( \sigma = 0 \), the linear equations for the coefficients are simplify considerably, and since the range of inaction becomes \([0, m^*]\), there is one non-linear equation in one unknown, Appendix A.4 displays the relevant equations for this case.

We present a proposition showing that the limit of small and frequent jumps is identical to the case of continuous consumption.

**Proposition 3.** Consider the solution of the value function as \( z \to 0 \) and \( 0 < \gamma \equiv \lim_{z \to 0} z \times \kappa < \infty \). This solution coincides with the one without jumps, i.e. \( \kappa = z = 0 \) but with continuous consumption at the rate \( c + \gamma \).

The logic of the proof of this proposition is straightforward, so we only sketched the argument here. First, notice that path for the the cumulative consumption accounted for jumps \( zN(t) \) goes to \( \gamma t \) with probability one. Second, notice that the contribution of these jumps to the value function, given by \( \kappa(V(m - z) - V(m)) \) when \( m > z \) can be written as \( \kappa (V(m) - V'(m)z + o(z)) \). Assuming that we can permute the limit of the derivative with the derivative of the limit, we obtain that in the limit this contribution of this terms goes to \( -\gamma V'(m) \), a term analogous to the contribution from \( c \). The contribution of the segment \( m > z \) is negligible as \( z \) goes to zero.

This result can be useful to make contact with the Austrian diary data, which obviously
is discrete in nature. The issue is not whether consumption transactions occurs as discrete events or not, which of course they do. The previous result says that small frequent purchases can be approximated using the continuous model. The issue is whether the continuous consumption model is a good approximation given the observed size of purchases. Thus if the purchases using cash are small and frequent, the model with continuous path may be a good idealization. On the other hand, intuitively, a model with infrequent and large purchases, will be the most different case, a set up to which we will turn in Section 3.3.

We finish this section with a brief comment on the optimality of the class of trigger policies considered here. First, in the case where $\sigma = 0$, it is easy to show that the ergodic distribution of $m$ lies in $[0, m^*]$, whose interior contains the inaction region. Second, in the case where there are no jumps, $\sigma > 0$ and $p = 0$, it has been shown, for instance in Constantinides and Richard (1978), that trigger policies of this type are optimal. The extension to the case of $p > 0$ should be relatively straightforward. The third, more subtle case is the combination of jumps (so that $\kappa > 0, z > 0$) and a brownian motion (so that $\sigma > 0$) for cumulated net cash consumption. The potential complication comes about when there are discrete changes in the unregulated state, i.e. discrete changes in $m$ in our case. This case has been studied in discrete time, finite but arbitrary horizon by Neave (1970). He has shown that the decision rule will, in general, have an inaction region close to the optimal return point, that outside the inaction region there is a set of intervals where either adjustment or inaction is optimal, and that for large values there is an open ended interval for which adjustment is optimal. Bar-Ilan (1990) has also produced a counterexample in the case of two periods two jumps, one up and one down. More recently Chen and Simchi-Levi (2009) have a slightly fuller characterization of this case and an analysis of a more general case. Hence, the issue in our continuous time model with jumps, which make the model mathematically very close to a discrete time model, is whether there could be several inaction and adjustment regions. Thus, while the form of the optimal policy for a model where the state follows the sum of a diffusion and a more general jump component, such as in the specification of Bar-Ilan,
Perry, and Stadje (2004) has not been characterized, our set-up is special enough so that the decision rules, in the ergodic set for \( m \), are the special form considered above. The features that make our problem special are that the state the jumps are all downwards and of the same size (i.e. \( z > 0 \)) and that the state is non-negative. In our case, if the state reaches \( m^{**} \) then it is controlled to be set at \( m^* \). Importantly, since the jumps in net cash consumption are all downwards, the state can only reach \( m^{**} \) at time \( t = \tau \) if it was below, but very close, at times arbitrary close to \( \tau \). On the other side, the boundary at \( m = 0 \) follows from non-negativity of cash and from the fact that the period return function attains its minimum at \( m = 0 \). Thus, the value of \( m^{**} \) is defined as the smallest strictly positive value of the state for which adjustment is optimal.\(^5\)

### 3.2 Solution for the case of no Brownian shocks

In this section we concentrate in the special case of the model where the net cumulated cash consumption is the sum of a deterministic constant consumption per unit of time and random jumps, i.e. we set the brownian component to have zero variance or \( \sigma = 0 \). We concentrate in this case for two reasons. The first is that our data sets for Italy and Austria focuses on households, which tends to have non-negative uses of cash. Related to this, notice that in this case in the model there will be no deposits, and only withdrawals: the inaction region is given by \([0, m^*]\), and cash inside this region only moves down, either at a constant rate per unit of time, or with jumps of size \( z \) and frequency \( \kappa \). Consistent with this behavior notice that Table 1 shows that Italian households make very few deposits relative to withdrawals: for 2002 the average ratio of the number of deposits to the number of withdrawals is less than 1%. If we concentrate on self-employed households, whose cash movements presumably resemble

---

\(^5\) We can write a discrete time version of our model in the notation of Neave (1970) and Chen and Simchi-Levi (2009) as follows. To simplify we write the version with \( p = 0 \). Let \( \Delta \) be the length of the time period. The period return function is \( l(m) = +\infty \) if \( m < 0 \) and otherwise \( l(m) = \Delta R m \). The i.i.d. process for unregulated cash, \( \xi = -\Delta c + \sqrt{\Delta \sigma} s - z dN \), where \( s \) is a symmetric binomial with zero mean and standard deviation one, where \( dN = 1 \) with probability \( \kappa \Delta \) and zero otherwise, and where \( dN \) and \( s \) are independent. The discount factor is \( \gamma = 1/(1 + r \Delta) \). The cost function has \( K = Q = b \) and no proportional cost, \( k = q = 0 \). In term of their notation we have, as we let \( \Delta \downarrow 0 \), the decision rules satisfy: \( U = T = m^* \), \( 0 = t = t^+ \), and \( u^- = m^{**} \).
more the ones for forms, i.e. having decreases and increases in cash, we see that deposits are more frequent than for the rest of the households, with an average ratio of number of deposits to the number of withdrawals of about 6%. Yet, deposits are quite infrequent, so we concentrate in the model with $\sigma = 0$ which generates the extreme case of no deposits. The second reason is that this model is simpler.

In this section we display expression for the form of the value function and for cash management statistics of interest for the case of $\sigma = 0$ and $\pi = 0$. Besides the value function $V(m)$ we also define the functions $M(m), w(m), \overline{m}(m)$ and $n(m)$ as the expected discounted (at rate $\rho$) integral of the respective quantities (cash balances, withdrawals size, cash-at-withdrawal and deposit indicator), conditional on the current value of $m$, where cash holding follow the law of motion that corresponds to the optimal decision rule of the model with $\sigma = \pi = 0$ and optimal return $m^*$. If there is no free withdrawal opportunity, cash balances evolve as

$$dm(t) = -c \, dt - z \, dN(t) \text{ for } m(t) \in (0, m^*)$$

where $N(t)$ is the counter of a Poisson process with arrival at the rate $\kappa$. If there is a free adjustment, which occurs with Poisson arrival rate $p$ per unit of time. We let $m_t$ the cash at the time of a withdrawal, and $w_t$ the amount of the withdrawal. In an instant an adjustment (withdrawal) can happen in either of the following three cases:i) $m(t)$ reaches zero in which case, $m_t = 0$ and $w_t = m^*$, ii) a jump in cash consumption occurs when $m(t) < z$, then $m_t = m(t)$ and $w(t) = z + m^* - m(t)$, iii) a free withdrawal opportunity takes place, then $m_t = m(t)$ and $w_t = m^* - m(t)$. We use the functions $M(m), w(m), \overline{m}(m)$ and $n(m)$ to compute the expected value under the invariant distribution of the money holdings $M$, the average withdrawal size $W$, the average cash holdings at the time of withdrawal $\frac{M}{M}$, and the average number of withdrawals per unit of time $n$. We define the unconditional expected values $(M, W, \overline{M}, n)$ by multiplying the expected discounted value by the discount rate $\rho$. This adjustment converts these quantities in a flow. For these cash holding statistics we take the discount rate to zero, to obtain the corresponding expected value under the invariant
distribution of the process for \( \{m(t)\} \), as explained below. Alternatively, we can also find the limits of \( M, W, \underline{M} \) and \( n \), by finding the invariant distribution of \( m \), say \( h \), and the expected number of withdrawals, \( n \) and using them to define the remaining statistics (\( M, \underline{M} \) and \( W \)). We do so in the appendix Appendix A.8, but the derivation and calculations of \( n \) and \( h \) are more involved and specialized. Instead here we derive then in a unified manner, without solving for \( h \) directly finding an expression for several moments of interest. In practice we implement the limit numerically, by solving the equations for a small value of \( \rho \). In particular let:

\[
M (m) = \mathbb{E} \left[ \rho \int_0^\infty e^{-\rho t} m(t) \, dt \mid m_0 = m \right]
\]

\[
w (m) = \mathbb{E} \left[ \rho \sum_{j=0}^{\infty} e^{-\rho \tau_j} (m(\tau_j^+) - m(\tau_j^-) + z I_{\tau_j}) \mid m_0 = m \right]
\]

\[
\underline{m} (m) = \mathbb{E} \left[ \rho \sum_{j=0}^{\infty} e^{-\rho \tau_j} m(\tau_j^-) \mid m_0 = m \right]
\]

\[
n (m) = \mathbb{E} \left[ \rho \sum_{j=0}^{\infty} e^{-\rho \tau_j} \mid m_0 = m \right]
\]

\[
V (m) = \frac{1}{\rho} [R M (m) + b (n (m) - p)]
\]

where \( \tau_j \) are the times at which a withdrawal happens (which may coincide with a free withdrawal opportunity or with a jump in consumption or not) and where \( I_t \) is an indicator that a cash consumption jump has occurred at time \( \tau_j \) when \( m(\tau_j^-) \leq z \). The expectations are taken with respect to the process for \( \{m(t)\} \) generated by equation (12). Notice that the value function \( V \) is the sum of the expected discounted cost of holding cash plus the expected discounted cost of the adjustments. The factor \( 1/\rho \) corrects the flow nature of the definitions for \( M \) and \( n \). Since the adjustments \( n \) include those that are free, the last terms subtracts the expected discounted value of them. In the case of the value function we let \( \rho = r \). For
the cash holding statistics we are interested in

\[ M = \lim_{\rho \to 0} M(m), \quad w = \lim_{\rho \to 0} w(m), \quad m = \lim_{\rho \to 0} m(m), \quad n = \lim_{\rho \to 0} n(m). \]

As in implicit in the notation, in the limit as \( \rho \downarrow 0 \) the functions do not depend on \( m \). Note that \( w \) is the expected value of the total amount of withdrawals during a period on length 1, and hence average withdrawal size is

\[ W = \frac{w}{n}. \]

Likewise, \( m \) is the expected value of the total amount of cash at the time of a withdrawal in a period of length 1, and hence the average cash at the time of a withdrawal is

\[ \underline{M} = \frac{m}{n}. \]

These functions satisfy the following system of ODE equations, which, in order to simplify the solution, we only write for the case of \( \pi = 0 \). The logic for them is the same as the one for the value function in the general case discussed in Section 3.1. We let \( j^* \) be the smallest integer for which \( m \leq (j^* + 1)z \). Thus, all these functions will be defined in segments of the form \([zj, z(j + 1)]\). For \( m \in [0, z] \) we have:

\[ (\rho + \kappa + \rho) F_0 (m) = \rho \nu_0 m + \rho \alpha_0 - F_0' (m) c + (\kappa + \rho) F^* \]

for suitable choices of the constants \( \nu_0 \) and \( \alpha_0 \) (see the Appendix A.6 for details). We follow the notational convention that the function evaluated right after hitting the barrier \( m^* \), i.e. at \( m(t^+) = m^* \), is denoted with a *, say for instance \( M(m^*) = M^* \). For \( m \in [zj, z(j + 1)] \) for \( j = 1, 2, ..., j^* \)

\[ (\rho + \kappa + \rho) F_j (m) = \rho \nu m + \rho \alpha - F_j' (m) c + \kappa F_{j-1} (m - z) + p F^* , \]

20
for some suitable choices of $\alpha$ and $\nu$ (see Appendix A.6 for details). Continuity at $m = z_j$ for $j = 1, \ldots, j^*$

$$F_j(z_j) = F_{j-1}(z_j)$$

for $j = 1, 2, \ldots, j^*$. The conditions at $m = 0$ are

$$F_0(0) = \rho \alpha^* + F^*,$$

for a suitable choice of $\alpha^*$ (see the Appendix A.6 for details). Now we can write the solution for $F$ as a function of $m^*, \nu, \nu_0, \alpha, \alpha_0$ and $\alpha^*$.

**Proposition 4.** Assume that $c > 0$, $\rho > 0$, and $\rho + p + \kappa > 0$. The ODE-DDE for $F$ has the following solution. Let $m^*$ and $\theta \equiv (\nu, \nu_0, \alpha, \alpha_0, \alpha^*)$ be given. Define $j^*$ as the smallest integer so that $(j^* + 1)z \geq m^*$. Then $F_j(\cdot; m^*, \theta) : [z_j, z(j + 1)] \to \mathbb{R}$ has the form:

$$F_j(m; m^*, \theta) = G_j + S_j(m - z_j) + \sum_{i=0}^{j^*} H_{ji} e^{\lambda (m - z_j)} (m - z_j)^i$$

where given the constant $\lambda$:

$$\lambda = \frac{\rho + \kappa + p}{-c},$$

and the values for $G_j$, $S_j$, $H_{ij}$ for $j = 1, \ldots, j^*$ and $i = 1, \ldots, j$ solve a block recursive system of linear equations described in the proof.

We use this general set-up to develop a non-linear equation to find the value of the optimal return point $m^*$. This equation reflect that $m^*$ is chosen optimally, and hence it must satisfy that $V'(m^*) = V'_{j^*}(m^*) = 0$. This can be written as:

$$0 = S_{j^*} + \sum_{i=0}^{j^*} H_{j^*i} e^{\lambda (m^* - z_{j^*})} i (m^* - z_{j^*})^{i-1} + \lambda \sum_{i=0}^{j^*} H_{j^*i} e^{\lambda (m^* - z_{j^*})} (m^* - z_{j^*})^i$$
For future reference we note that in the model with $\sigma = 0$, we have the accounting identity
$W n = e$, where $e \equiv c + \kappa z$. Using the definition of $n_{BT} = e/(2M)$ for the Baumol-Tobin benchmark, the previous accounting identity we have that

$$\frac{n}{n_{BT}} \equiv \frac{2nM}{e} = \frac{2}{W/M}. \quad (13)$$

Thus, the implications of the theory for values of $W/M$ higher (smaller) than 2, the value that corresponds to Baumol-Tobin, are equivalent to values of $n/n_{BT}$ smaller (higher) than one.

### 3.3 The case of infrequent large purchases

We continue with the analysis of the case of no inflation and no Brownian component, i.e. $\pi = \sigma = 0$. Furthermore we solve the model and the cash holding statistics $M, W, n, M$ for a configuration of parameters that corresponds to the case of small $\kappa$, (i.e. smaller than $\kappa^*$) in the deterministic model of Section 2 and large value of $z$. While the simple and start implications are only for the case of large and infrequent purchases, we found it instructive for two reasons: first, given the result in Proposition 3, the case with $z$ large and $\kappa$ small, and presents interesting differences compared to the problem with continuous cumulated net cash expenditures. Second, we argue below that the parameters values for which this case applies, i.e. the size and frequency of the large purchases, seem to be empirically relevant for households in Austria.

Abusing notation, we let $V (m; m', p, \kappa, z, c)$ denote the value function of the model analyzed in Section 3.2 when cash is $m$ and the return threshold for cash is $m'$ for the parameters $(p, \kappa, z, c)$. We also let $m^* (p, \kappa, z, c)$ the value of the optimal threshold for these parameters, and let $M (p, \kappa, z, c), W (p, \kappa, z, c), M (p, \kappa, z, c)$ and $n (p, \kappa, z, c)$ be the corresponding cash-
management statistics, described in Section 3.2. For future reference we let

\[ V^* (m'; p, \kappa, z, c) \equiv V (m'; m', p, \kappa, z, c) \]  

the value of following a policy with return threshold \( m' \) when cash is at this value. Recall that at the optimal threshold value \( V^* (m^*(p, \kappa, z, c); p, \kappa, z, c) \) is the smallest value of the value function.

The next proposition compares the model without large purchases (i.e. no jumps in \( C(t) \)) but with possible free withdrawal opportunities, with a model with large infrequent large purchases (or jumps) and possible also free withdrawal opportunities. We note that if \( z = 0 \) or \( \kappa = 0 \) the model with no jumps corresponds to a version of Baumol-Tobin where there are \( p \) free withdrawal opportunities per unit of time. We have characterized the solution of that model and estimated it for a cross section of Italian households in Alvarez and Lippi (2009).

The free withdrawal opportunities of that model imply that, relative to the prediction in Baumol-Tobin, agents makes more withdrawals (say \( n/(c/2M) \equiv n/n_{BT} > 1 \)) and they are smaller in size (say, \( W/M < 2 \)). Also differently to Baumol-Tobin, that model implies that in average agents withdraw when they have strictly positive real balances, i.e. \( M > 0 \). We use the notation \( m^*(p', 0, 0, c) \) to denote the optimal return threshold for the model with no jumps, with a rate of \( p' \) free adjustment opportunities per unit of time, and with consumption at a constant rate \( c \).

**Proposition 5.** Assume that \( \pi = \sigma = 0, c > 0, p \geq 0, b/R > 0, z \) and \( r > 0 \). There
exists $\kappa^* > 0$ such that for any $\kappa < \kappa^*$ and $z > m^* (p + \kappa, 0, 0, c)$ we have:

$$m^* (p, \kappa, z, c) = m^* (p + \kappa, 0, 0, c),$$
$$V^* (m^* (p, \kappa, z, c); p, \kappa, z, c) = V^* (m^* (p + \kappa, 0, 0, c), p + \kappa, 0, 0, c) + \frac{\kappa b}{r},$$
$$M (p, \kappa, z, c) = M (p + \kappa, 0, 0, c),$$
$$W (p, \kappa, z, c) = W (p + \kappa, 0, 0, c) + \frac{\kappa z}{n},$$
$$\overline{M} (p, \kappa, z, c) = \overline{M} (p + \kappa, 0, 0, c),$$
$$n (p, \kappa, z, c) = n (p + \kappa, 0, 0, c).$$

Moreover, if $z' > z$ the conclusion holds for the same value of $\kappa^*$.

Part of the proof of the proposition is straightforward. In particular, if it is optimal to set $m^* (p, \kappa, z, c) < z$, then the value of the threshold equal the one in a model with no jumps, i.e. with $z = 0$, but with $p + \kappa$ free adjustment opportunities, i.e. $m^* (p + \kappa, 0, 0, c)$. In other words, it is always a local minimum to set the return threshold equal to $m^* (p + \kappa, 0, 0, c)$. In the case of no jumps at each free withdrawal opportunity, cash balances jump to $m^*$ right after the adjustment. The arrival of a free adjustment opportunity is similar to the event in which an agent must withdraw because of a jump in their consumption. In both cases, after adjustment cash balances goes to $m^*$, and also both events are assumed to occur independently of the cash balances. The difference is that in the free withdrawal opportunity the agent does it because it saves the cost $b$, while with the jump of consumption the agent does it because of the binding non-negativity of consumption. For the exact equivalence we require that the rate at which free adjustment opportunities arrives is $p + \kappa$, so the event that ends up in an adjustment when cash is strictly positive is equally likely. The two value functions differ only in a constant, the present value of the cost saved by the free withdrawal opportunities. If follows that the average cash holding, average cash at withdrawals and average number of adjustment are equal. The average withdrawal size differs, because in the case where there is a jump, there is an extra withdrawal of size $z$, which happens in the $\kappa$
of withdrawals $n$. Finally, to show that if $m^*(p + \kappa, 0, 0, c) < z$ the it is optimal to set this value for for the thresholds in the case of jumps, provided that $\kappa$ is small, is more subtle. While the proof of this statement is a bit more involved, its logic is analogous to the one of the determinist model of Section 2.

To understand one of the hypothesis in Proposition 5, it is useful to give a characterization of $m^*_0 \equiv m^*(p + \kappa, 0, 0, c)$. In Alvarez and Lippi (2009) we show that it given by: $m^*_0 = \varphi(b/(Rc), p + r)$, which is the unique positive solution to:

$$
\exp\left((r + p + \kappa)\frac{m^*_0}{c}\right) = \frac{m^*_0}{c}(r + p + \kappa) + 1 + (r + p + \kappa)^2\frac{b}{cR}
$$

or equivalently

$$
\frac{b}{cR} = \left(\frac{m^*_0}{c}\right)^2\left[1 + \sum_{i=1}^{\infty}\left(\frac{m^*_0}{c}(r + p + \kappa)\right)^i \frac{1}{(2 + i)!}\right]
$$

(15)

Clearly $m^*_0$ is a strictly increasing function of $b/R$, which goes from 0 to $\infty$ as $b/R$ varies in the same range, and it is decreasing in $p$. The limit as $r + p + \kappa \downarrow 0$ is the familia Baumol-Tobin expression $m^*_0/c = \sqrt{2b/(cR)}$. Finally, $m^*_0$ is increasing in $c$ with and elasticity between $1/2$ and 1. Thus, for a fixed $z$, one of the hypothesis of Proposition 5 holds, for a small enough fixed cost relative to opportunity cost, i.e. small $b/R$. Also, since $m^*_0$ is decreasing in $p + \kappa + r$, thus $\sqrt{2bc/R} \geq c \varphi(b/(Rc), r + p + \kappa)$. Hence a sufficient conditions for $z \geq m^*(p + \kappa, 0, 0, c)$ is that $z \geq \sqrt{2bc/R} \geq c$.

A direct implication of Proposition 5 is that data on $M, W, \underline{M}, n$ and $e$, assuming that the hypothesis $m^* < z$ holds, can not identify $\kappa, p$ and $z$ separately. It can only identify $\kappa z$ and $\kappa + p$. For any pair of $\kappa + p$ and $z \kappa$ as well as the rest of the parameters that are consistent with $M, W, \underline{M}, n$ and $e$, so are the pairs $\kappa' + p'$ and $z' \kappa'$, with $\kappa + p = \kappa' + p'$ and $z \kappa = z' \kappa'$, with $z' > z$ and $\kappa' < \kappa$.

We now describe a condition that the ratios $W/M$ and $\underline{M}/M$ must satisfy to be consistent with the behaviour described in the hypothesis of Proposition 5. Additionally, we describe how to identify $\kappa z/e$ and $\kappa + p$, as long as the previous condition is met. To do so, we first describe all the implications of the observable statistics $M, W, \underline{M}, n$ and $e$ for the model’s
parameters under the hypothesis of Proposition 5.

**Proposition 6.** Assume that the hypothesis of Proposition 5 hold. Then model implies the following relationship between the five observable statistics \((M, W, \overline{M}, n, c + \kappa z)\) and four structural parameters \((c, \kappa z, p + \kappa, b/R)\), and the threshold \(m^*\):

\[
\begin{align*}
n &= \frac{\kappa + p}{1 - \exp\left(-\left(\kappa + p\right)\frac{m^*}{c}\right)}, \\
W + \overline{M} &= m^* + \frac{\kappa}{n}z, \\
n \frac{M}{\overline{M}} &= \kappa + p, \\
c + \kappa z &= nW, \\
\frac{m^*}{c} &= \varphi\left(\frac{b/c}{R}, r + p + \kappa\right).
\end{align*}
\]

The proof of this proposition is straightforward. **Equation (16)** follows from using the fundamental theorem of renewal theory equals the reciprocal of the expected time between successive withdrawals. Times between withdrawals is distributed as a truncated exponential with parameter \(\kappa + p\). The times are truncated at time \(\bar{t} \equiv m^*/c\), the time it will take to deplete money holdings with continuous consumption. **Equation (17)** follows from taking expected values of the cash flows at time of a withdrawal. It states that on average after a withdrawal an agent has balances \(m^*\), which is the sum of the average cash at the time of withdrawal \((\overline{M})\) and the withdrawal size \((W)\) net of the fraction \(\kappa/n\) of the withdrawals where a consumption jump of size \(z\) is financed. **Equation (18)** follows from computing the average cash holdings at the time of an adjustment. A fraction \(1 - (p + \kappa)/n\) of the withdrawals the agent has reached zero cash holdings at the time of a withdrawal. A fraction \((p + \kappa)/n\) of the withdrawals the agents has strictly positive cash holdings, and since the occurrence of these adjustment are independent of the level of cash holdings, in these cases the agents has the average cash holdings. **Equation (19)** is simply the budget constraint. Up to here,
the implications follow from the form of the optimal decision rules.\footnote{One can also add an expression that computes the value of average value of \( M \), using \( n, m^* \) and parameters, namely \( M/c = (n m^*/c - 1)/(\kappa + p) \). Yet, this equation is implied by equations (16)-(19).} Finally, equation (20), already presented and discussed in equation (15) ensures that the value of the threshold \( m^* \) is optimal. Using Proposition 5 we have replaced here \( p \) by \( p + \kappa \).

We use the expressions in Proposition 6 to solve for the fraction of expenditures that corresponds to jumps, i.e. \( \kappa z/(c + \kappa z) \), the value of \( m^* \), which gives a lower bound to \( z \) in order for the proposition to apply, and the value of \( p + \kappa \). We have:

\[
\frac{\kappa z}{\kappa z + c} = 1 + \frac{1}{W/M} \left( \frac{M/M}{\log(1-M/M)} + 1 \right), \tag{21}
\]

\[
\kappa \leq p + \kappa = n \frac{M}{M} \leq n, \tag{22}
\]

\[
z \geq m^* = M \left( \frac{\log(1-M/M)}{1 + \log(1-M/M)} \right). \tag{23}
\]

The next proposition summarizes the implications for \( W/M \) and \( M/M \) of \( \gamma, b/(cR) \) and \( p + \kappa \) in the case where \( m^* \leq z \). To simplify the expressions we take the limit as \( r \downarrow 0 \), in which case, given \( \gamma \) all the other other parameters combine in a single index to explain the effects on \( W/M \) and \( M/M \).

**Proposition 7.** Let \( r \downarrow 0 \) and assume that \( z, b/(cR), \) and \( p + \kappa \) are such that \( m^* \leq z \). Define \( \gamma \equiv \kappa z/(z\kappa + c) \). Then

\[
\frac{W}{M} = \frac{1}{1 - \gamma} \omega \left( \frac{b/c}{R} \frac{(p + \kappa)^2}{M/M} \right) \text{ and } \frac{M}{M} = \mu \left( \frac{b/c}{R} \frac{(p + \kappa)^2}{M/M} \right) \tag{24}
\]

where \( \omega : \mathbb{R}_+ \to [0,2] \) is strictly decreasing and \( \mu : \mathbb{R}_+ \to [0,1] \) is strictly increasing. The functions \( \omega \) and \( \mu \) depends on no other parameters, and are given in the appendix.

For completeness we consider the case where the large purchases are frequent. As in the deterministic case, if for a given size of the purchases \( z \), the frequency \( \kappa \) is high enough,
Figure 1: $\rho V^*(m^*; m^*)$: conditional value function plotted against threshold $m^*$ at $m = m^*$.

Proposition 8. Let $z > 0$, $c \geq 0$, $p \geq 0$ and $b/R > 0$. There exist $\kappa^{**}(z)$, which is increasing in $z$, and $r^{**} > 0$ such that for any $\kappa > \kappa^{**}(z)$ and $r < r^{**}$, the optimal threshold satisfies $m^*(p, \kappa, z, c) > z$.

The logic of this proposition is the same as in the deterministic case, as well as the increasing nature of the threshold $\kappa^{**}$. We believe that the assumption that $r$ is small is not required for the results, but it simplifies the constructive aspect of the proof.

Figure 1 helps understand the different cases covered in Proposition 5 and Proposition 8.
depending on the value of $\kappa$. In this figure we have plot the value of following a policy characterized by threshold $m^*$, evaluated at $m = m^*$ for different values of this threshold. The optimal policy can be obtained for the value of $m^*$ that minimizes $V(m^*, m^*)$. Interestingly this function is not single peaked. This example shows that simple adding the boundary condition $V'(m^*; m^*) = 0$ does not insure the optimality of the given policy. Note the difference with models without the jump component, such as Constantinides (1978), where a verification theorem states that any function that solves the relevant ode and boundary conditions is a solution of the problem. The parameter values considered for this figure correspond to an “intermediate” value for $\kappa$ and $z$ for which the optimal threshold has $m^* \approx 35 > z = 20$. Note that while setting $m^* = m^*(p + \kappa, 0, 0, c) = 15 < z = 20$, so that every jump would triggers a withdrawal, is a local minimum of $V(m^*; m^*)$ but it is not the global minimum. In other words, the values for this example do not satisfied the hypothesis that $\kappa < \kappa^*$ of Proposition 5. Notice also that setting $m^* = 51$, which is the optimal for the case of no jumps, but a larger continuous consumption equal to $c + \kappa z$ is also not the optimal for the case with jumps, but it is also close to a local minimum.

4 Empirical Applications

In this section we describe two different applications of the ideas behind our model. The first case applies the model to currency management using survey and diary data from Austrian households. In this case the expenditures refer to those paid with currency. The second case applies the model to the management of liquid assets for Italian investors, using a panel data set of bank accounts. In this case we take liquid asset to be a concept similar to M2, and the expenditures to be durables vs non-durables.

While in this paper we focus on the implication of these large purchases for cash management, a related interesting issue is what explain the choice of means of payments of different purchases, especially as related to the size of the purchases. There is a related literature, both
empirical and theoretical on this issue, see for instance Whitesell (1989), Bounie, Francois, and Houy (2007) and Mooslechner, Stix, and Wagner (2006). The problem of the choice of means of payment differ across economies. In particular we conjecture that for developing economies and less developed countries, where alternative to cash are less prevalent, even more people will be paying for large purchases using cash, and hence the issues discussed in this paper are even more relevant. Preliminary work using panel data from rural Thailand in Alvarez, Pawsutipaisit, and Townsend (2011) supports this hypothesis. Additionally we think that the issue of lumpy net cash consumption models may be relevant for firms cash balances. In particular, we think that firms that go through large cash inflows and outflows should behave differently, such as the those related to purchases and sales of assets which need to be done using liquid assets.

4.1 Currency Management of Austrian and Italian Households

Table 1 display some baseline statistics coming from two households surveys, one from Italy and one from Austria. We present all statistics splitting the sample on those that have an ATM card and those that do not. We split the sample in those with and without ATM cards as a way to control for the “opposite” type of behavior that can be caused by “free withdrawals” opportunities captured by the parameter $p$, as explained above. We display the mean across households (individuals for Austria) of several statistics: share of consumption that is paid using currency for both countries, the average amount of currency held $M$, the average amount of currency held at the time of a withdrawal $\bar{M}$, the average size of a withdrawal $W$, the average size of deposits $D$, the number of deposits per year $n_D$ and number of withdrawals per year $n$. Several of the statistics are computed as ratios, which we think help in interpreting them in terms of the model. For instance, we use $M/e$, the average money to cash consumption, at daily frequencies, $W/M$ the average withdrawal size to average currency held, the ratio of the average size of deposits to withdrawals $W/D$, the ratio of $n$ to the deposits implied by accounting identity $Wn = e$ and the assumption that
### Table 1: Currency management statistics in Italy and Austria

<table>
<thead>
<tr>
<th></th>
<th>ATM Card</th>
<th>Italy (2002)</th>
<th>Austria (2005)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expenditure share paid w. currency</td>
<td>w/o</td>
<td>0.65(^a)</td>
<td>0.96(^a)</td>
</tr>
<tr>
<td></td>
<td>w.</td>
<td>0.52(^a)</td>
<td>0.73(^a)</td>
</tr>
<tr>
<td>Currency: $M/e \ (e/\text{per day})$</td>
<td>w/o</td>
<td>17(^b)</td>
<td>15(^B)</td>
</tr>
<tr>
<td></td>
<td>w.</td>
<td>13(^b)</td>
<td>15(^B)</td>
</tr>
<tr>
<td>$M$ per Household</td>
<td>w/o</td>
<td>410(^c)</td>
<td>332(^C)</td>
</tr>
<tr>
<td></td>
<td>w.</td>
<td>330(^c)</td>
<td>206(^C)</td>
</tr>
<tr>
<td>Currency at withdrawals(^d): $M/M$</td>
<td>w/o</td>
<td>0.46</td>
<td>0.22</td>
</tr>
<tr>
<td></td>
<td>w.</td>
<td>0.41</td>
<td>0.26</td>
</tr>
<tr>
<td>Withdrawal(^e): $W/M$</td>
<td>w/o</td>
<td>2.0</td>
<td>2.4</td>
</tr>
<tr>
<td></td>
<td>w.</td>
<td>1.3</td>
<td>1.6</td>
</tr>
<tr>
<td>Withdrawal / Deposit(^h,i): $W/D$</td>
<td>w/o</td>
<td>0.68 [0.53]</td>
<td>n.a.</td>
</tr>
<tr>
<td></td>
<td>w.</td>
<td>1.13 [0.84]</td>
<td>n.a.</td>
</tr>
<tr>
<td># of withdrawals: $n$ (per year)(^f)</td>
<td>w/o</td>
<td>23</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td>w.</td>
<td>58</td>
<td>68</td>
</tr>
<tr>
<td>Normalized: $\frac{n}{n_{BT}} = \frac{n}{e/(2M)} \ (e/\text{per year})$</td>
<td>w/o</td>
<td>1.7</td>
<td>1.4</td>
</tr>
<tr>
<td></td>
<td>w.</td>
<td>3.9</td>
<td>5.4</td>
</tr>
<tr>
<td># of deposits / # withdrawals(^h): $n_D/n$</td>
<td>w.</td>
<td>0.007 [0.058]</td>
<td>n.a.</td>
</tr>
<tr>
<td>Fraction of households with $W/M &gt; 2$</td>
<td>w/o</td>
<td>0.25</td>
<td>0.29</td>
</tr>
<tr>
<td></td>
<td>w.</td>
<td>0.13</td>
<td>0.19</td>
</tr>
<tr>
<td>Fraction of households with $\frac{n}{n_{BT}} \equiv \frac{n}{e/(2M)} &lt; 1$</td>
<td>w/o</td>
<td>0.50</td>
<td>0.57</td>
</tr>
<tr>
<td></td>
<td>w.</td>
<td>0.19</td>
<td>0.31</td>
</tr>
<tr>
<td># of observations</td>
<td>w/o</td>
<td>2,275(^g)</td>
<td>153(^c)</td>
</tr>
<tr>
<td></td>
<td>w.</td>
<td>3,729(^g)</td>
<td>895(^G)</td>
</tr>
</tbody>
</table>

Entries are sample means. The unit of observation is the household for Italy; for Austria, the subject of the survey are men and women 14 years and older, not households. Only households with a checking account (both Austria and Italy) and whose head is not self-employed (Italy) are included, with the exception of data in square brackets [·], which are computed only for households whose head is self-employed (approximately 17% of all households with a bank account).

Notes for Italian data: Source: Bank of Italy - *Survey of Household Income and Wealth.*

- \(^a\)Ratio of expenditures paid with currency to total expenditures (durables, non-durables and services).
- \(^b\)Average currency held by the household during the year divided by daily expenditures paid with currency. \(^c\)In 2004 euros. \(^d\)Average currency at the time of withdrawal as a ratio to average currency. \(^e\)Average withdrawal during the year as a ratio to average currency. \(^f\)The entries with $n = 0$ are coded as missing values. \(^g\)Number of respondents for whom the currency and the currency-consumption data are available in each survey. Data on withdrawals are supplied by a smaller number of respondents. \(^h\)Sample average over 1993-2000. \(^i\)Computed for households reporting $D > 0$.

Notes for Austrian data: Source: Austrian National Bank - *OeNB.*

- \(^A\)Numerator and denominator of the ratio are based on transactions collected in a diary kept for 7 days. The diary excludes automatic payments and likely misses large transactions (a broader measurement would produce smaller values for the ratio).
- \(^B\)Average currency carried by the individual (sum of currency with them and currency available at home; items 18 and 18a in questionnaire), divided by daily expenditures paid with currency. Respondents keep a large fraction of currency balances at home; The average of the ratio of currency at home to total currency held is about 60%. \(^C\)In 2005 Euros. \(^G\)Number of respondents with a bank deposit account and non-zero values for $M, W, e, n$. This accounts for about 87% of the sample.
withdrawals occur when cash it zero and replenish that amount, $W = 2M$, so $n_{BT} = e/(2M)$. The statistics display in Table 1 show that in several dimensions in which Austrian and Italian households cash management is similar.

Motivated by the outcomes of the special case analyzed in Proposition 5 and Proposition 6 of the model with $\sigma = 0$, we compute several statistics that intent to measure the degree of large infrequent cash purchases for individuals in Austria. We split the sample on those with and without ATM cards, because, at least using only the cash management statistics $M, W, M, e$, the model does not identify separately $p$ and $\kappa$. Yet the value of $p$ should be related (positively) with the density and availability of ATMs. Thus, the split between those with and without ATM cards serves as a way to “control” for the value of $p$. Most of the data comes from the survey and a diary of Austrian households described in Mooslechner, Stix, and Wagner (2006). The survey contains a series of retrospective questions on cash management, as well as on method and pattern of purchases. The diary, asked individuals to record all purchases made in the following week.

In the first panel of Table 3 we use data from a retrospective question in the survey about actual large purchases paid with cash made during the last month. This question ask about purchases that are larger than 400 euros each. We split the data on those individual with an ATM card and those without. We remark that the threshold of 400 euros for large purchases, chosen independently by the survey designers, end up being a reasonable approximation for the value of $z$ satisfying $z > m^*$. While, $m^*$ is not directly measurable, in that case we have seen in equation (23) that $m^*/M$ can be measure based on currency at withdrawals $M/M$. This can be seen in the row the records the mean and median of an statistic we labelled as $z/M$. This is the average large purchase (for those that made at least one last month) divided by the average currency holding. Given our characterization of $z/m^*$ in equation (23), we find this supportive evidence for the threshold of 400 euros. We note that the fractions of individuals with at least one large purchase in the last month is similar among those with

\[\text{In Alvarez and Lippi (2009) we estimate the model with } \kappa = z = 0 \text{ for Italian households and verify this implication.}\]
<table>
<thead>
<tr>
<th>Table 2: Cash Management and Large Purchases in Austria</th>
</tr>
</thead>
<tbody>
<tr>
<td>% persons that use cash for large purchase</td>
</tr>
<tr>
<td>All (1048 Obs)</td>
</tr>
<tr>
<td>Withdrawal to Money: $W/M$</td>
</tr>
<tr>
<td> </td>
</tr>
<tr>
<td># withdrawals relative to BT$^b$: $n/n_{BT}$</td>
</tr>
<tr>
<td>Normalized cash at withdrawals $nM/M^c$</td>
</tr>
<tr>
<td>% persons that do not use cash for large purchase</td>
</tr>
<tr>
<td>All (1048 Obs)</td>
</tr>
<tr>
<td>Withdrawal to Money: $W/M$</td>
</tr>
<tr>
<td> </td>
</tr>
<tr>
<td># withdrawals relative to BT$^b$: $n/n_{BT}$</td>
</tr>
<tr>
<td>Normalized cash at withdrawals $nM/M^c$</td>
</tr>
</tbody>
</table>


- $^a$ Based on a question about how individual usually paid for items that cost more than 400 euros. Two options are available, either currency or other payment methods. Total number of respondents is 1048.
- $^b$ # of withdrawals $n$ relative to Baumol-Tobin benchmark, $n_{BT} = e/(2M)$ Based on a diary of all transactions during a week. This the week is right after the month corresponding to the question on large transactions above.
- $^c$ the variable $nM/M$ is the the product of the number of withdrawals $n$ and the ratio of the average cash at the time of withdrawal, $M$ to the average cash holdings.
and without ATM card, about 23% per month each. Taken as face value, this corresponds to a value of \(\kappa = 2.2\) at annual frequencies. We also note that among those that have had a purchase, the average, 1st quartile (not shown in the table) and median size, all relative to their average cash holdings, are larger for those with ATM cards. This difference is mainly due mostly to the fact that those with ATM cards report to have in average much less cash, about 60% of those without cards, as seen in Table 1.

Table 3: Statistics on the lumpiness of cash purchases in Austria

<table>
<thead>
<tr>
<th></th>
<th>with ATM card</th>
<th>without ATM card</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large purchases (&gt; 400 euros) in 1 month (^a)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>% persons with no large purchase</td>
<td>84%</td>
<td>82%</td>
</tr>
<tr>
<td>large purchase size / money (^b): (z/M) for (z &gt; 0)</td>
<td>11.9, 4.0</td>
<td>11.4, 2.4</td>
</tr>
<tr>
<td>All purchases (diary data) (^c)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Purchase size: average / median (^d): (e_a/e_m)</td>
<td>1.6, 1.3</td>
<td>1.6, 1.3</td>
</tr>
<tr>
<td>Number of purchases per week</td>
<td>10, 9</td>
<td>10, 9</td>
</tr>
<tr>
<td>Average purchase size / income (^e): (e_a/y)</td>
<td>2.2, 1.4</td>
<td>4.4, 2.3</td>
</tr>
</tbody>
</table>

Based on a survey and diary of Austrian individuals described in Mooslechner, Stix, and Wagner (2006). - \(^a\) These statistics are based on a question about the purchases larger than 400 euros each made during the last month. - \(^b\) Computed among those with purchases during the month. - \(^c\) Based on a diary of all transactions during a week. This the week is right after the month corresponding to the question on large transactions above. - \(^d\) The ratio uses the average and median transaction size paid with currency for each individual during the week of the diary. - \(^e\) The value of the average purchase made with currency (\(e_a\)) divided by the income per month of the individual (\(y\)).

In the second panel of Table 3 we use the diary expenditures data, taken during a week. We first note that, given how infrequent are the large purchases in cash per month, there is no enough information in a week of expenditures to measure heterogeneity across individuals by using the 400 euros threshold. Instead we use other measures that we hope capture the lumpiness from the diary data. From this diary we report statistics for the ratio of the average cash purchase to the median cash purchase, both for those with and without ATM cards. Additionally, we report statistics for the number of purchases made with cash during a week.
for those with and without an ATM card. We found that for both measures the statistics are very similar for those that have ATM cards and those that do not.

Table 4: Currency as the usual payment method for purchases of different size

<table>
<thead>
<tr>
<th></th>
<th>All (1048 Obs)</th>
<th>w/ATM card (895 Obs.)</th>
<th>w/o ATM card (153 Obs.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>% Individuals that use currency(a)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Purchases between ([0, 10]) euros</td>
<td>95.6</td>
<td>94.9</td>
<td>100.0</td>
</tr>
<tr>
<td>Purchases between ([11, 30]) euros</td>
<td>83.3</td>
<td>80.6</td>
<td>99.4</td>
</tr>
<tr>
<td>Purchases between ([30, 50]) euros</td>
<td>70.4</td>
<td>65.6</td>
<td>98.7</td>
</tr>
<tr>
<td>Purchases between ([51, 100]) euros</td>
<td>58.1</td>
<td>51.5</td>
<td>97.7</td>
</tr>
<tr>
<td>Purchases between ([100, 200]) euros</td>
<td>47.6</td>
<td>39.2</td>
<td>97.7</td>
</tr>
<tr>
<td>Purchases between ([201, 400]) euros</td>
<td>43.2</td>
<td>24.2</td>
<td>96.1</td>
</tr>
<tr>
<td>Purchases between ([401, \infty]) euros</td>
<td>45.9</td>
<td>27.4</td>
<td>96.1</td>
</tr>
</tbody>
</table>

- \(a\) Percentage of individuals that answer that currency is the usual method of payments for purchases for each different size. The alternatives are currency or other method. Based on 1048 responses for each purchase size.

Table 4 and Table 2 use a different type of information on lumpy cash purchases. These tables use the answer to the survey question of which is the usual method of payments for purchases of different sizes. The answers are currency or other method. Table 4 shows the small size purchases are made by almost all individuals using currency, but that less than half of the individuals use currency as the usual means of payments for the purchases of 400 euros or more. These statistics are presented separately for those with ATM cards and for those without, which show a clear difference. Almost all individuals without an ATM card use cash as the usual payments regardless of the size of the purchases. Table 2 displays three cash management statistics for those individuals that use cash for large purchases (top panel) and for those that do not (bottom panel). These statistics are also presented separately for those with ATM cards. We display the ratio of the average size of withdrawals to the average cash holdings, \(W/M\), as well as the ratio of the number of withdrawals relative to the one implied by Baumol-Tobin, \(n/n_{BT}\). Recall that, if the accounting identity \(Wn = e\) is satisfied, the product of these two statistic should be 2. We present both statistics because this accounting
does not hold individual by individual. We interpret the source of this discrepancy as to be caused by measurement error. Consistent with this interpretation, we find that the patterns of violations of this identity are such that symmetric and are centered around zero. We interpret the model as having implications in the comparison of individuals for which currency is not the usual means of payment for large purchases with those that don’t. For those that use currency, everything else the same, we expect $W/M$ to be larger, and $n/n_{BT}$ to be smaller. Comparing the top and the bottom panels of Table 2 we find some support for this prediction for all the individuals and for those with ATM cards. We don’t find this for those without ATM cards, but we notice that among them there are only 6 individuals that did not use cash as the usual payment methods, which makes the precision of the estimates potentially more problematic. With respect to $nM/M$, recall that in the simple model equals $p + \kappa$. Consistent with our hypothesis than those that have ATM cards have a higher value of $p$. On the other hand, as explain when we discussed the frequency of large purchases in Table 3, the value of $\kappa$ seems to be similar for both type of individuals, and small relative to $nM/M$.

Now we report the pattern of cash-management statistics vs proxys for the lumpiness of purchases measured from the diary data. We first present cash management statistics pooling all the data on Austrian individuals. We find some patterns that are broadly consistent with what the model predicts as a consequence of variation on $z$ across agents. We interpret the ratio of the average purchase size to its media $e_a/e_m$ as a measure of $z$. We find that $e_a/e_m$ is negatively correlated with $n/n_{BT}$ and positively with $W/M$ (see Figure 3). We did not find any correlation between $e_a/e_m$ and $M/M$, which is consistent with the idea that it is determined by variation on $p + \kappa$, instead of $z$, and dominated by the variation on $p$.

In Figure 3 we repeat the same patterns using the ratio of average purchase size to monthly income, $e_a/y$. In this case we don’t find a positive correlation of this proxy for lumpiness and the ratio $W/M$. This is consistent with the fact that the positive correlation between $e_a/e_m$ and $W/M$ displayed in Figure 3 was in large part due to the difference between individuals with and without ATM cards.
Figure 2: Austria: the inventory model statistics vs. $e_a/e_m$

**$n/n_{BT}$**

- Regr. coeff. $-1.01$, t-stat $-10.0$

**$W/M$**

- Regr. coeff. $0.17$, t-stat $2.4$

**$M/M$**

- Regr. coeff. $0.001$, t-stat $0.001$

Note: Log scale.
Figure 3: Austria: the inventory model statistics vs. $e_a/y$

$n/n_BT$

Regr. coeff. $-0.59$, $t$-stat $-11.8$

$W/M$

Regr. coeff. $-0.04$, $t$-stat $-1.3$

$M/M$

Regr. coeff. $-0.09$, $t$-stat $-2.0$

Note: Log scale.
We now turn to a comparison across ATM ownership groups of the cash management statistics using other measures of lumpiness for cash consumption. Figure 4 compares these statistics, individual by individual within each group. In particular, the figure plots the normalized number of withdrawals, \( n/n_{BT} \) against the normalized average purchase size \( e_a/y \). The negative correlation displayed is consistent with what the model predicts as a consequence of variation on \( z \) across agents.

Another proxy for lumpy cash purchases \( z \) is the ratio between the individual’s average purchase size \( (e) \) divided by the individual’s median purchase size \( (e_m) \) based on the Austrian diary data. In Figure 5 we also find a negative correlation between this proxy and the ration \( n/n_{BT} \).

### 4.2 Management of Liquid Asset by Italian Investors

TO BE WRITTEN
Figure 5: Austria: normalized withdrawal frequency vs. ratio of average to median purchase size

<table>
<thead>
<tr>
<th>w/o ATM Card</th>
<th>w. ATM card</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Graph" /></td>
<td><img src="image" alt="Graph" /></td>
</tr>
<tr>
<td>Regr. coeff. $-1.03$, t-stat $-3.7$</td>
<td>Regr. coeff. $-0.97$, t-stat $-9.1$</td>
</tr>
</tbody>
</table>

Note: Log scale. The vertical axis reports the normalized number of withdrawals $n/n_{BT}$.

References


A Proofs

A.1 Proof of Proposition 1.

Proof. We consider two possible patterns for cash management, depending on the relative frequencies of withdrawals and jumps in cumulative consumption. In the first, the agent makes a withdrawal every \( j/\kappa \) units of time, where \( j \geq 1 \). In this case there are \( j \) jumps in cumulated consumption between withdrawals. In the second, the agent withdraws every \( 1/(i \kappa) \) units of time, where \( i \geq 1 \). In this case, there are \( i \) withdrawals between jumps in cumulative consumption.

Case I: withdrawals every \( j/\kappa \) units of time (\( j \) jumps between withdrawals)

We consider policies where agents make a withdrawal every \( j/\kappa \) units of time, and thus there are \( j \geq 1 \) jumps in cumulated consumption between two withdrawals. Thus the number of withdrawals per unit of time, the size of the withdrawal, and the average cash balances per unit of time:

\[
\begin{align*}
n(j) &= \frac{\kappa}{j}, \\
W(j) &= c \frac{j}{\kappa} + z j = c \frac{j}{n} + \frac{\kappa z}{n}, \\
M(j) &= \frac{1}{2} c \frac{j}{\kappa} + \frac{1}{2} z (j-1) = \frac{(c + z \kappa)}{2 n} - \frac{1}{2} z.
\end{align*}
\]

for and integer \( j \geq 1 \). the number of withdrawals per unit of time \( n \) is the reciprocal of the time between withdrawals. The expression for withdrawal size \( W \) accounts for flow of \( c \) during the time period of length \( j/\kappa \). In this case the cash balances between two withdrawals decrease continuously with consumption \( c \) and discontinuously at times that are multiplies of \( 1/\kappa \) by the amount \( z \). The first part of the expression for average cash balances \( M \) contains the contribution to required average cash balances from the continuous consumption \( c \), an expression identical to the one in Baumol-Tobin. The second part contains the contribution due to the “jumps”, or discontinuous consumption. The last part of the expression contains only \( (j-1) \) because the last consumption jump is financed by the corresponding withdrawal.

The objective function is then

\[
C_I \equiv \min_{j \geq 1} R M(j) + b n(j) \equiv \min_{\kappa/n \in \mathbb{N}^+} \frac{R (c + z \kappa)}{2 n} + b n - \frac{R z}{2}
\]

Notice that, except for the constraint on \( n \) the relevant problem to decide \( n \) is as in Baumol-Tobin, but with total consumption equal to \( c + \kappa z \). Thus the optimal decision rule and objective function, ignoring integer constraint on \( j \), but imposing that \( j \geq 1 \), or equivalently \( n \leq \kappa \), can be written as

If \( 2 b \kappa^2 \geq R (c + z \kappa) \) \( \implies n_I = \sqrt{\frac{R (c + z \kappa)}{2 b}} \) and \( C_I = \sqrt{2 R b (c + \kappa z)} - \frac{R z}{2} \)

If \( 2 b \kappa^2 < R (c + z \kappa) \) \( \implies n_I = \kappa \) and \( C_I = \frac{R c}{2} + b \kappa \)
The ratio of average withdrawal to average money stock is given by:

\[
\frac{W}{M} = 2 \frac{c + \kappa z}{c + z(\kappa - n)} \geq 2.
\]

If \(2b \kappa^2 < R(c + z\kappa)\) \(\implies\) \(1 < \frac{W/M}{2} = \frac{c + \kappa z}{c + z(\kappa - n)} < \frac{c + \kappa z}{c}\)

If \(2b \kappa^2 \geq R(c + z\kappa)\) \(\implies\) \(\frac{W/M}{2} = \frac{c + \kappa z}{c}\)

**Case II: withdrawals every \(1/(i\kappa)\) units of time (\(i\) withdrawals between jumps)**

Consider policies where agents make \(i\) withdrawals in a period of length \(1/\kappa\), where \(i\) is an integer. Thus, the time between withdrawals is \(1/(i\kappa)\). In this case the number of withdrawals per unit of time is \(i\kappa\). The size of withdrawals varies, since 1 every \(i\) withdrawals includes the amount for a consumption jump. Finally, the average average money holdings are identical to those of the Baumol Tobin model where consumption is given by the continuous component only. Hence we have:

\[
\begin{align*}
n(i) &= i \kappa, \\
W(i) &= \frac{z}{i} + \frac{c}{i \kappa} = \frac{\kappa z}{n} + \frac{c}{n}, \\
M(i) &= \frac{c}{2i \kappa} = \frac{c}{2n}.
\end{align*}
\]

The agent solves:

\[
C_{II} = \min_{i \geq 1} R M(i) + b n(i) = \min_{n/\kappa \in I^+} R \left[ \frac{c}{2n} \right] + b n.
\]

Notice that, except for the constraint on \(n\) the relevant problem to decide \(n\) is as in Baumol-Tobin, but with total consumption equal to \(c\). Thus the optimal decision rule and objective function, ignoring the integer constraints on \(n\), but imposing that \(n \geq \kappa\) can be written as

\[
\begin{align*}
\text{If } 2b \kappa^2 &\leq R c \implies n_{II} = \sqrt{\frac{Rc}{2b}} \text{ and } C_{II} = \sqrt{2Rb} c \\
\text{If } 2b \kappa^2 &> R c \implies n_{II} = \kappa \text{ and } C_{II} = \frac{Rc}{2} + b\kappa
\end{align*}
\]

The ratio of average withdrawal to average money stock is given by:

\[
\frac{W}{M} = 2 \frac{c + \kappa z}{c} \geq 2.
\]

**Optimal Policy: combining Case I and Case II**

Now we can obtain the decision rule, combining the case where there are multiple jumps between withdrawals (case I) and there are multiple withdrawals between jumps (case II).
We first find an expression for the thresholds \((\kappa^*, \kappa^{**}(z))\) for which the constraint on \(i \geq 1\) and \(j \geq 1\) binds, for case II and I respectively.

\[
\kappa^* = \sqrt{\frac{Rc}{2b}} \leq \kappa^{**}(z) = \frac{Rz + \sqrt{(Rz)^2 + 8bRc}}{4b}.
\]

Note that \(\kappa^* = \kappa^{**}(0)\) and that \(\kappa^{**}\) is strictly increasing in \(z\). We can then write:

If \(\kappa \leq \kappa^* \implies n = n_{II} = \sqrt{\frac{Rc}{2b}}\); and \(C = C_{II} = \sqrt{2Rb}\)

If \(\kappa^* < \kappa < \kappa^{**}(z) \implies n = \kappa\) and \(C = C_I = C_{II} = \frac{Rc}{2} + b\kappa\)

If \(\kappa \geq \kappa^{**}(z) \implies n = n_{I} = \sqrt{\frac{R(c + z\kappa)}{2b}}\); and \(C = C_I = \sqrt{2Rb(c + \kappa z)} - \frac{Rz}{2}\)

For simplicity, in the characterization of the optimal policies, we disregarded the constraint that either \(j\) or \(i\) are integers. A necessary and sufficient conditions to disregard that constraint is the following. Define \(u\) as follows:

\[
u \equiv \max \left\{ \sqrt{\frac{Rc}{2b}} \kappa, \sqrt{\frac{\kappa^2}{2b}} \right\}.
\]

The condition is that if \(u > 1\), then \(u\) is an integer. In this case the constraint that \(n\) is an integer is not binding.

### A.2 Proof of Proposition 2.

**Proof.** Step 1: Deriving a system of ODE’s

Taking as given the values of \(m^*\) and \(m^{**}\). Using these values, we split the range of inaction for \(V\), given by \([0, m^{**}]\), into \(J\) intervals. The first \(J - 1\) intervals are of width \(z\) and are given by \([jz, (j + 1)z]\) for \(j = 0, 1, ..., J - 1\). The last interval is given by \([Jz, \min\{m^{**}, (J + 1)z\}]\).

We also define \(j^*\) as the smallest integer such that \(z(j^* + 1) \geq m^*,\) so that \(zj^* m^* < z(j^* + 1).\)

We index the solution of each of the ODE’s by \(j\). We start with \(V_0 : [0z] \rightarrow \mathbb{R}\) which solves the linear second order (first order if \(\sigma = 0\)) ODE:

\[
(r + p + \kappa)V_0(m) = Rm + (p + \kappa)V^* + \kappa b + V'_0(m)(-c - \pi m) + \frac{\sigma^2}{2}V''_0(m)
\]

for \(0 \leq m \leq z\). For \(1 \leq j \leq J\), taking as given the function \(V_{j-1}(\cdot)\), we have the following linear second order (first order if \(\sigma = 0\)) ODE for \(V_j : [jz, \min\{(j + 1)z, m^{**}\}] \rightarrow \mathbb{R}\):

\[
(r + p + \kappa)V_j(m) = Rm + pV^* + \kappa V_{j-1}(m - z) + V'_j(m)(-c - \pi m) + \frac{\sigma^2}{2}V''_j(m)
\]

for \(j = 1, ..., J - 1\). We also have the following value matching at \(m = 0, m = m^{**}\) and
\[ m = m^* : \]

\[
V_0(0) = V^* + b, \quad (A-3)
\]
\[
V_{j-1}(m^*) = V^* + b, \quad (A-4)
\]
\[
V_j(m^*) = V^* . \quad (A-5)
\]

Since \( V \) is twice differentiable for \( \sigma > 0 \), (once if \( \sigma = 0 \)) we require that

\[ V_j(z_j) = V_{j-1}(z_j) \text{ and if } \sigma > 0, \ V'_j(z_j) = V'_{j-1}(z_j), \]

for \( j = 1, 2, \ldots, J - 1 \). Notice that for \( \sigma > 0 \), if the \( V_j \) solve their corresponding ODE, then these pair of equalities implies that the second derivatives of \( V_j \) and \( V_{j-1} \) agree at each point for \( j \geq 1 \). This can be shown, recursively, starting from \( j = 1 \). Instead if \( \sigma = 0 \), if \( V_j(z_j) = V_{j-1}(z_j) \), then the first derivative agrees on these points.

Step 2: Solving the system of ODEs

Up to here, given \((m^*, m^{**})\) we have a system of second order (first order if \( \sigma = 0 \)) linear differential equations, with exactly as many boundary conditions to find a unique solution, which we denote by \( V(\cdot; m^*, m^{**}) \) on the range \([0, m^{**}]\). The function \( V(\cdot; m^*, m^{**}) \) is the value of following a policy indexed by \((m^*, m^{**})\). In particular, for any given \( V^* \), there is a two parameter family (one parameter if \( \sigma = 0 \)) that solves \( V_0 \) in \([0, z]\). Thus, fixing \( V^* \) and these two parameters (one if \( \sigma = 0 \)) we can use \( V_0 \) to solve for \( V_1 \) in the range, \([z, 2z]\). These second order (first order if \( \sigma = 0 \)) ODE uses the two (one if \( \sigma = 0 \)) boundary conditions: \( V_0(z) = V_1(z) \) and (if \( \sigma > 0 \)) \( V'_0(z) = V'_1(z) \). We continue recursively, solving for \( V_j \) on \([jz, (j+1)z]\) for \( j = 1, \ldots, J - 1 \), using the previously found solution for \( V_{j-1} \) on \([(j-1)z, jz]\), each time using the two (one if \( \sigma = 0 \)) boundary conditions \( V_{j-1}(jz) = V_j(jz) \) and (if \( \sigma > 0 \)) \( V'_{j-1}(jz) = V'_j(jz) \). At the end of this procedure we have functions \( V_0, V_1, \ldots, V_{J-1} \) depending on \( V^* \) and two parameters (one if \( \sigma > 0 \)). We can solve for these three numbers (two if \( \sigma = 0 \)) imposing the value matching conditions equations (A-3)-(A-4)-(A-5).

We note that in the case in which \( \sigma = 0 \), the homogeneous linear second order (first order if \( \sigma = 0 \)) ODE have constant coefficients. Hence the solution of the homogenous is given by the linear combination of two exponential functions. The solution of the non-homogenous solution is given by sum of the product of each of the solution of the homogenous and other function. This can be computed recursively, starting from \( j = 0 \). This gives the following solution in equation (9).

Step 3: Deriving the linear equations for the coefficients of the solution of equation (9)

First we will take as given the two values of \( B_{00}^k \) for \( k = 1, 2 \) and develop a system of equation for \( \{A_j, D_j\} \) and the remaining \( \{B_{j,i}^k\} \).

The variable \( V^* \) can be eliminated of the system using the value matching at \( m = 0 \), equation (A-3) and the form of \( V_0 \), namely

\[ V^* = A_0 + \sum_{k=1,2} B_{0,0}^k - b . \quad (A-6) \]

We solve for the coefficients of \( V_0 \) on the constant and multiplying \( m \) on both sides of the
ODE We have

\[ D_0 = \frac{R}{r + p + \kappa}, \quad A_0 = \frac{(p + \kappa)V^* + \kappa b - c D_0}{r + p + \kappa} \] (A-7)

Replacing the conjectured form of \( V_j \) in equation (9) on both sides of the ODE equation (A.2) for \( j = 1, 2, \ldots, J - 1 \):

\[
(r + p + \kappa) \left( A_j + D_j (m - zj) + \sum_{k=1,2} \sum_{i=0}^{j} B_{j,i}^k e^{\lambda_k (m-zj)} (m-zj)^i \right) 
= Rm + pV^* - c \left( D_j + \sum_{k=1,2} \sum_{i=0}^{j} B_{j,i}^k e^{\lambda_k (m-zj)} (\lambda_k (m-zj)^i + i (m-zj)^{i-1}) \right) + \kappa \left( A_{j-1} + D_{j-1} (m - zj) + \sum_{k=1,2} \sum_{i=0}^{j-1} B_{j-1,i}^k e^{\lambda_k (m-zj)} (m-zj)^i \right) + \frac{\sigma^2}{2} \left( \sum_{k=1,2} \sum_{i=0}^{j} B_{j,i}^k e^{\lambda_k (m-zj)} \left( \lambda_k^2 (m-zj)^i + 2\lambda_k i (m-zj)^{i-1} + i(i-1) (m-zj)^{i-2} \right) \right)

For \( 1 \leq j \leq J - 1 \) : matching the constant and coefficients on \( m \) on both sides of the ODE for \( V_j \) equation (A-8) we have:

\[
D_j = \frac{R}{r + p + \kappa} + \frac{\kappa}{r + p + \kappa} D_{j-1}, \quad A_j = \frac{pV^* - c D_j - \kappa zj D_{j-1} + \kappa A_{j-1}}{r + p + \kappa} + D_j zj .
\] (A-9)

Thus, using equation (A-6), equation (A-7) and equation (A-9) we can solve for all \( \{D_j, A_j\}_{j=1,\ldots,J-1} \) as functions of \( B_{00}^k \) for \( k = 1, 2 \).

Now we match the coefficients of the terms involving \( e^{\lambda_k (m-zj)} (m-zj)^i \) in both sides of equation (A-8), the ODE for \( V_j \). Fixing an ODE \( j = 1, \ldots, J - 1 \), we have coefficients for \( i = 0, 1, \ldots, j \). Matching the coefficient for \( e^{\lambda_k (m-zj)} (m-zj)^j \) gives no additional restrictions, given the expression for \( \lambda_k \). The coefficient for \( e^{\lambda_k (m-zj)} (m-zj)^{j-1} \) gives the following difference equation for \( B_{j,j,j-1}^k \):

\[
B_{j,j}^k (c - \sigma^2 \lambda_k) = \kappa B_{j-1,j-1}^k \quad \text{for} \quad j = 1, \ldots, J - 1, \quad k = 1, 2 .
\] (A-10)

Thus using equation (A-6) we can solve for \( \{B_{j,j}^k\}_{j=1,\ldots,J-1} \) given \( B_{00}^k \) for \( k = 1, 2 \).

Likewise matching the coefficients for \( e^{\lambda_k (m-zj)} (m-zj)^i \) for \( i = 0, 1, \ldots, j - 2 \), canceling some terms due to the expression for \( \lambda_k \), gives

\[
(c - \sigma^2 \lambda_k) (i+1) B_{j,i+1}^k = \kappa B_{j-1,i+1}^k + \frac{\sigma^2}{2} B_{j,i+2}^k (i+1) (i+2) \quad \text{for} \quad j = 2, \ldots, J - 1, \quad i = 0, \ldots, j - 2, \quad k = 1, 2 .
\] (A-11)

Imposing that the level and the first derivative of the functions \( V_{j-1} \) and \( V_j \) agree when
evaluated at $z_j$ we obtain:

$$A_j + D_j + \sum_{k=1,2} B_{j,0}^k = A_{j-1} + D_{j-1}z + \sum_{k=1,2} \sum_{i=0}^{j-1} B_{j-1,i}^k e^{\lambda_k z} z^i$$  \hspace{1cm} (A-12)

$$D_j + \sum_{k=1,2} B_{j,0}^k \lambda_k = D_{j-1} + \sum_{k=1,2} \sum_{i=0}^{j-1} B_{j-1,i}^k \lambda_k z^i [\lambda_k z^i + i z^{i-1}] ,$$ \hspace{1cm} (A-13)

for $j = 1, 2, \ldots, J - 1$.

We will now solve for $\{B_{j,i}^k\}$ for $j = 1, \ldots, J - 1$ and $i = 0, \ldots, j - 1$, for each $k = 1, 2$. First, we can use equation (A-12) and equation (A-13) for $j = 1$ to solve for $B_{1,0}^k$. Using these values and $\{B_{2,2}^k\}$ we can use equation (A-11) and equation (A-12)-equation (A-13) for $j = 2$ to solve for $\{B_{2,1}^k, B_{2,2}^k, 0\}$. In general, on one hand, fixing $k = 1, 2$ and $j = 2, \ldots, J - 1$ if $\{B_{j-1,i}^k\}_{i=0, \ldots, j-2}$ are known, equation (A-11) can be solve for $\{B_{j,i}^k\}_{i=0, \ldots, j-1}$ using $B_{j,j}^k$ as a known boundary condition. On the other hand, we can use equation (A-12) and equation (A-13) for $j = 2, \ldots, J - 1$ to obtain two extra linear equations for $\{B_{j,0}^k\}_{i=0, \ldots, j, k=1,2}$, given the values of $\{B_{j-1,i}^k\}_{i=0, \ldots, j-1}$.

At this point we have solved for $\{D_j, A_j, B_{j,i}^k\}$ as functions of $B_{0,0}^k$. We can now solve for $B_{0,0}^k$ using two more linear equations: the value at $V_{j^*}$ at $m^*$ and of $V_{J-1}$ at $m^{**}$ has to satisfy:

$$V^* = A_{J^*} + D_{J^*} (m^* - z_{J^*}) + \sum_{k=1,2} \sum_{i=0}^{J^*} B_{J^*,i}^k e^{\lambda_k (m^* - z_{J^*})} (m^* - z_{J^*})^i ,$$ \hspace{1cm} (A-14)

$$V^* + b = A_{J-1} + D_{J-1} (m^{**} - z(J - 1)) + \sum_{k=1,2} \sum_{i=0}^{J-1} B_{J-1,i}^k e^{\lambda_k (m^{**} - z(J - 1))} (m^{**} - z(J - 1))^i$$ \hspace{1cm} (A-15)

We note that while $V^*$ appears in these equation, it can be replaced by using equation (A-6) in terms of $A_0, B_{0,0}^k$.

### A.3 Proof of Proposition 3.

**Proof.** Consider the Bellman equation for the case of continuous consumption at the rate $\gamma + c$ and no jumps, so $z = 0$. If $0 < m < m^{**}$ it reads:

$$(r + p)V(m) = Rm + pV^* - V'(m)(c + \gamma).$$

Instead he Bellman equation for the case of jumps, when $z < m < m^{**}$ is

$$(r + p)V(m; z) = Rm + pV^*(z) - V'(m; z)c + \kappa (V(m; z) - V(m - z; z)).$$

Where the $z$ as a second argument is included to emphasize that it is the problem with jumps. We want to show that as $z \to 0$ both Bellman equations coincide, which is the same
than showing that:

$$V'(m)\gamma = \lim_{z \to 0} \kappa (V(m; z) - V(m - z; z)).$$

This instead holds by writing $V(m - z; z) = V(m; z) - V'(m; z)z + o(z)$. Taking the limit as $z \to 0$ and using that $\kappa z = \gamma$. QED

A.4 Solution of the Bellman equation for the $\pi = \sigma = 0$ case.

In the case the linear system described in the proof of Proposition 2 further simplifies. In particular we have that there is only one root $\lambda$ so we omit $k$ from all the expression.

Since $\sigma = 0$, the range of inaction is $[0, m^*]$. Let $j^*$ the smallest integer such that $(j^* + 1)z \geq m^*$. The value function in each segment $V_j : [z_j, z(j+1)] \to \mathbb{R}$ for $j = 0, ..., j^*$:

$$V_j(m) = A_j + D_j (m - z - j) + \exp(\lambda (m - z j)) \sum_{i=0}^{j} B_{j,i} (m - z j)^i$$

where the constants $V^*, \lambda, A_j, D_j$ and $B_{j,i}$ satisfy the following set of linear equations:

$$\lambda = \frac{r + p + \kappa - c}{-c}, \quad D_0 = \frac{R}{(r + p + \kappa)},$$

$$(r + p + \kappa) A_0 = (p + \kappa) V^* + \kappa b - \frac{c}{r + p + \kappa}, \quad B_{0,0} = b + V^* - A_0.$$

and for $j = 0, 1, ..., j^* - 1$:

$$D_{j+1} = -\frac{1}{\lambda} \left[ \frac{R}{c} + \frac{\kappa}{c} D_j \right], \quad A_{j+1} = \frac{1}{\lambda} \left( D_{j+1} - \frac{p V^*}{c} - \frac{\kappa}{c} [A_j - D_j z (j + 1)] \right) + D_{j+1} z (j + 1),$$

$$B_{j+1,0} = A_j + D_j z + e^{\lambda z} \sum_{i=0}^{j} B_{j,i} z^i - A_{j+1}, \quad B_{j+1,i+1} = \frac{1}{i + 1} \frac{\kappa}{c} B_{j,i} \text{ for } i = 0, 1, 2, ..., j$$

$$V^* = A_{j^*} + D_{j^*} (m^* - z - j^*) + e^{\lambda (m^* - z j^*)} \sum_{i=0}^{j^*} B_{j^*,i} (m^* - z j^*)^i$$

Finally, the optimality of the threshold $m^*$ implies that:

$$0 = D_{j^*} + e^{\lambda (m^* - z j^*)} \left[ \sum_{i=0}^{j^*} B_{j^*,i} \left( \lambda (m^* - z (j^*))^i + i (m^* - z j^*)^{i-1} \right) \right]$$

(A-16)
A.5 Proof of Proposition 4

First we describe the system of linear equations that the coefficients for $G_j$, $S_j$, $H_{ij}$ for $j = 1, \ldots, j^*$ and $i = 1, \ldots, j$ and $F^*$ solve:

$$S_0 = \frac{\rho}{\rho + \kappa + p} \nu_0,$$

$$G_0 = \frac{\rho}{\rho + \kappa + p} \alpha_0 - \frac{c}{\rho + \kappa + p} S_0 + \frac{(\kappa + p)}{\rho + \kappa + p} F^*,$$

$$H_{00} = \rho \alpha^* + F^* - G_0,$$

for $j = 0, 1, 2, \ldots, j^* - 1$:

$$S_{j+1} = \frac{\rho}{\rho + \kappa + p} \nu + \frac{\kappa}{\rho + \kappa + p} S_j,$$

$$G_{j+1} = \frac{\rho \alpha + p F^*}{\rho + \kappa + p} + \frac{\kappa}{\rho + \kappa + p} [G_j - S_j z (j + 1)] - \frac{c}{\rho + \kappa + p} S_{j+1} + S_{j+1} z (j + 1),$$

$$H_{j+1,0} = G_j + S_j z + \sum_{i=0}^{j} H_{ji} e^{\lambda z} z^i - G_{j+1}$$

where

$$H_{j+1, i} = \frac{1}{\lambda} \frac{\kappa}{i} H_{ji-1}$$

for $i = 1, 2, \ldots, j + 1$, and

$$F^* = G_j^* + S_j^* (m^* - z j^*) + \sum_{i=0}^{j} H_{j+i} e^{\lambda(m^* - z j^*)} (m^* - z j^*)^i$$

Second, we derive this equations. We do this in two steps.

Step I. Solution for $j = 0$.

We have

$$F_0 (m) = G_0 + S_0 m + H_{00} \exp (\lambda m)$$

and

$$F'_0 (m) = S_0 + \lambda H_{00} \exp (\lambda m)$$

or

$$(\rho + \kappa + p) \left[ G_0 + S_0 m + H_{00} e^{\lambda m} \right] = \rho \nu_0 m + \rho \alpha_0 - c \left[ S_0 + \lambda H_{00} e^{\lambda m} \right] + (\kappa + p) F^*$$

Now we solve for $\lambda$, $S_0$, $G_0$ and $H_{00}$.

For $\lambda$ we have:

$$\lambda = \frac{\rho + \kappa + p}{-c},$$

For $S_0$:

$$(\rho + \kappa + p) S_0 = \rho \nu_0$$
$$S_0 = \frac{\rho}{\rho + \kappa + p} \nu_0.$$  

For $G_0$

$$(\rho + \kappa + p) G_0 = \rho \, \alpha_0 - \kappa S_0 + (\kappa + p) F^*$$

or

$$G_0 = \frac{\rho}{\rho + \kappa + p} \alpha_0 - \frac{c}{\rho + \kappa + p} S_0 + \frac{(\kappa + p)}{\rho + \kappa + p} F^*$$

Using the boundary condition $F_0(0) = \rho \, \alpha^* + F^*$,

$$F_0(0) = G_0 + H_{00} = \rho \alpha^* + F^*$$

or

$$H_{00} = \rho \alpha^* + F^* - G_0.$$  

Step II. Solution for $1 \leq j \leq j^*$.  

We have

$$F'_j(m) = S_j + e^{\lambda (m-zj)} \sum_{i=0}^{j} H_{ji} \left[ \lambda (m-zj)^i + i \, (m-zj)^{i-1} \right]$$

so that

$$(\rho + \kappa + p) \left[ G_{j+1} + S_{j+1} (m-z(j+1)) + \sum_{i=0}^{j+1} H_{j+1,i} \exp (\lambda (m-z(j+1))) \, (m-z(j+1))^i \right]$$

$$= \rho \, \nu \, m + \rho \alpha + pF^*$$

$$+ \kappa \left[ G_j + S_j (m-z(j+1)) + \sum_{i=0}^{j} H_{ji} \exp (\lambda (m-z(j+1))) \, (m-z(j+1))^i \right]$$

$$- c \left[ S_{j+1} + \exp (\lambda (m-z(j+1))) \sum_{i=0}^{j+1} H_{j+1,i} \left[ \lambda (m-z(j+1))^i + i \, (m-z(j+1))^{i-1} \right] \right]$$

Matching the coefficients for $\exp (\lambda (m-z(j+1)))$ requires:

$$(\rho + \kappa + p) \sum_{i=0}^{j+1} H_{j+1,i} (m-z(j+1))^i = \kappa \sum_{i=0}^{j} H_{ji} (m-z(j+1))^i$$

$$- c \sum_{i=0}^{j+1} H_{j+1,i} \left[ \lambda (m-z(j+1))^i + i \, (m-z(j+1))^{i-1} \right]$$

51
and using the expression for $\lambda$

$$0 = \kappa \sum_{i=0}^{j} H_{ji} \ (m - z (j + 1))^i$$

$$- c \sum_{i=0}^{j+1} H_{j+1i} \ i \ (m - z (j + 1))^{i-1}$$

and matching the coefficients for $(m - z (j + 1))^{i-1}$:

$$0 = \kappa H_{j i-1} - c i H_{j+1 i}$$

so that

$$H_{j+1, i} = \frac{1}{c} \frac{\kappa}{i} H_{j, i-1}$$

for $i = 1, 2, ..., j + 1$.

Matching the coefficients of $m$:

$$(\rho + \kappa + p) S_{j+1} = \rho \nu + \kappa S_j$$

or

$$S_{j+1} = \frac{\rho}{\rho + \kappa + p} \nu + \frac{\kappa}{\rho + \kappa + p} S_j$$

Matching the coefficient for the constants:

$$(\rho + \kappa + p) [G_{j+1} - S_{j+1} z (j + 1)]$$

$$= \rho \alpha + p F^* + \kappa [G_j - S_j z (j + 1)] - c S_{j+1}$$

or

$$G_{j+1} = \frac{\rho \alpha + p F^*}{(\rho + \kappa + p)} + \frac{\kappa}{\rho + \kappa + p} [G_j - S_j z (j + 1)] - \frac{c}{(\rho + \kappa + p)} S_{j+1} + S_{j+1} z (j + 1)$$

Finally using the continuity at $zj$:

$$F_{j+1} (z (j + 1)) = G_{j+1} + S_{j+1} + \sum_{i=0}^{j+1} H_{j+1, i} (0)^i = G_{j+1} + H_{j+1, 0}$$

$$F_j (z (j + 1)) = G_j + S_j z + \sum_{i=0}^{j} H_{ji} \exp (\lambda z) (z)^i$$

so that

$$G_{j+1} + H_{j+1, 0} = G_j + S_j z + \sum_{i=0}^{j} H_{ji} \exp (\lambda z) (z)^i$$
or

\[ H_{j+1,0} = G_j + S_j \ z + \sum_{i=0}^{j} H_{j,i} \exp(\lambda z) (z)^i - G_{j+1}. \]

Finally we require that

\[ F^* = F_j^* (m^*) \text{ or } \]

\[ F^* = G_j^* + S_j^* (m^* - z^*) + \sum_{i=0}^{j^*} H_{j^*,i} \exp(\lambda (m^* - z^*)) (m^* - z^*)^i \]

QED

A.6 Bellman equations for \( V, M, \underline{M}, W, n \) and its coefficients

In the range \([0, z]\) we have:

\[
\begin{align*}
(\rho + \kappa + p) M_0 (m) &= \rho m - M'_0 (m) c + (\kappa + p) M^* \\
(\rho + \kappa + p) w_0 (m) &= \rho (\kappa + p) (m^* - m) + \rho k z - w'_0 (m) c + (\kappa + p) w^* \\
(\rho + \kappa + p) m_0 (m) &= \rho (\kappa + p) m - m'_0 (m) c + (\kappa + p) m^*_0 \\
(\rho + \kappa + p) n_0 (m) &= \rho (\kappa + p) - n'_0 (m) c + (\kappa + p) n^* \\
(\rho + \kappa + p) V_0 (m) &= R m + \kappa b - V'_0 (m) c + (\kappa + p) V^*
\end{align*}
\]

We follow the notational convention that the function evaluated right after hitting the barrier \( m^* \), i.e. at \( m(t^+) = m^* \), is denoted with a \(^*\), say for instance \( M(m^*) = M^* \). For \( m \in [z_j, z (j+1)] \) for \( j = 1, 2, \ldots, j^* \)

\[
\begin{align*}
(\rho + \kappa + p) M_j (m) &= \rho m - M'_j (m) c + \kappa M_{j-1} (m - z) + p M^* \\
(\rho + \kappa + p) w_j (m) &= \rho \rho (m^* - m) - w'_j (m) c + \kappa w_{j-1} (m - z) + p w^* \\
(\rho + \kappa + p) m_j (m) &= \rho \rho m - m'_j (m) c + \kappa m_{j-1} (m - z) + p m^* \\
(\rho + \kappa + p) n_j (m) &= \rho \rho n'_j (m) c + \kappa n_{j-1} (m - z) + p n^* \\
(\rho + \kappa + p) V_j (m) &= R m - V'_j (m) c + \kappa V_{j-1} (m - z) + p V^*
\end{align*}
\]

Continuity of these function across the segments gives:

\[
\begin{align*}
M_j (z_j) &= M_{j-1} (z_j), \\
w_j (z_j) &= w_{j-1} (z_j), \\
m_j (z_j) &= M_{j-1} (z_j), \\
n_j (z_j) &= n_{j-1} (z_j), \\
V_j (z_j) &= V_{j-1} (z_j),
\end{align*}
\]

53
The boundary conditions at $m = 0$ are

$$
M_0(0) = M^*,
$$
$$
w_0(0) = \rho m^* + w^*,
$$
$$
m_0(0) = m^*,
$$
$$
n_0(0) = \rho + n^*,
$$
$$
V_0(0) = V^* + b ,
$$

We display the expressions to map the equations for the general formulation $F$ to each of the 4 variables: $M$, $w$, $m$, and $n$. They are

$$
M_0 : \quad \nu_0 = 1, \quad \alpha_0 = 0,
$$
$$
w_0 : \quad \nu_0 = -(\kappa + p), \quad \alpha_0 = (\kappa + p)m^* + \kappa z,
$$
$$
m_0 : \quad \nu_0 = \kappa + p, \quad \alpha_0 = 0,
$$
$$
n_0 : \quad \nu_0 = 0, \quad \alpha_0 = (\kappa + p),
$$
$$
V_0 : \quad \nu_0 = R/\rho, \quad \alpha_0 = \kappa b/\rho,
$$

$$
M_j : \quad \nu = 1, \quad \alpha = 0,
$$
$$
w_j : \quad \nu = -p, \quad \alpha = pm^*,
$$
$$
m_j : \quad \nu = p, \quad \alpha = 0,
$$
$$
n_j : \quad \nu = 0, \quad \alpha = p,
$$
$$
V_j : \quad \nu = R/\rho, \quad \alpha = 0,
$$

and

$$
M_0 : \quad \alpha^* = 0 ,
$$
$$
w_0 : \quad \alpha^* = m^*,
$$
$$
m_0 : \quad \alpha^* = 0,
$$
$$
n_0 : \quad \alpha^* = 1,
$$
$$
V_0 : \quad \alpha^* = b/\rho.
$$

A.7 Proof of Proposition 5.

Proof. Consider the function $V(m; m', p, \kappa, z, c)$, the value of following a policy with an upper threshold $m'$ at the current value $m$. This function has been characterized fully in Proposition 4 for the appropriate setting of $\theta$. Recall that the solution of the agent’s problem $V(\cdot; m^*, p, \kappa, z, c)$ is minimized at $m^*$. Thus, we can find the optimal threshold $m^*$ by minimizing $V^*(m'; p, \kappa, z, c) \equiv V(m'; m^*, p, \kappa, z, c)$.

The condition that $m^*(p + \kappa, 0, 0, c) < z$ ensures that

$$
V^*(m', p, \kappa, z, c) \geq V^*(m^*(p + \kappa, 0, 0, c); p, \kappa, z, c) + \frac{\kappa b}{r}.
$$
for all \( m' \leq z \) every jump triggers a withdrawal, and hence, apart from the cost \( b \), the jumps behaves exactly as free withdrawal opportunities. Thus in this range the value function satisfies

\[
V'(m'; p, \kappa, c, z) = V^*(m', p + \kappa, 0, 0, c) + \kappa b / r .
\]

The value function \( V(\cdot; \cdot; 0, 0, 0, c) \) is the value function for the problem studied in Alvarez and Lippi (2009), with no jumps. Either using the results in Alvarez and Lippi (2009), or solving the relevant ODE and boundary condition in Proposition 4 for the case where \( m' < z \) we have the following explicit solution

\[
V^*(m', p + \kappa, 0, 0, c) = \frac{r + p + \kappa}{r} \left[ \frac{m' R}{r + p + \kappa} + b \frac{1}{1 - e^{-(r + p + \kappa)m'/c}} + b \right]
\]

This function is single peaked, attains its minimum at \( m^*/(p + \kappa, 0, 0, c) \) and is strictly increasing in \( m' \) for values \( m' > m^*/(p + \kappa, 0, 0, c) \).

The argument for values \( m' > z \) uses that for any \( \epsilon > 0 \), we can find \( \kappa^* \) such that for \( \kappa < \kappa^* \):

\[
|V^*(m^*(p, \kappa, z, c); p, \kappa, z, c) - V^*(m^*(p + \kappa, 0, 0, c); p + \kappa, 0, 0, c) - \frac{b\kappa}{r}| < \epsilon
\]

The result will follow by choosing \( \epsilon \) to be

\[
\epsilon(z) = V^*(z; p, \kappa, z, c) - V^*(m^*(p + \kappa, 0, 0, c); p + \kappa, 0, 0, c) - \kappa b / r .
\]

That \( \epsilon > 0 \) follows from the assumption that \( m^*(p + \kappa, 0, 0, c) < z \). Since \( V^*(z; p, \kappa, z, c) \) is increasing in \( z \), then so is \( \epsilon(z) \).

Now we show the required continuity. Consider a policy where, regardless of whether \( m < z \) or not, if a jump takes place the agent makes a withdrawal. The expected discounted cost of this policy equals \( V'(m'; p + \kappa, 0, 0, c) + \kappa b / r \). The first term is the expected discounted cost of financing a constant cash consumption of \( c \) and having free withdrawal opportunities at the rate \( p + \kappa \). The second term is the expected discounted cost of all the withdrawals that occur every time a jump occur. Since withdrawals occur even if \( m \geq z \) we have for all \( m' \):

\[
V^*(m'; p, \kappa, z, c) \leq V^*(m'; p + \kappa, 0, 0, c) + \kappa b / r .
\]

From the optimality of \( m^*(p + \kappa, 0, 0, c) \) and \( m^*(p, 0, 0, c) \) we have that for all \( m' \):

\[
V^*(m'; p + \kappa, 0, 0, c) \geq V^*(m^*(p + \kappa, 0, 0, c); p + \kappa, 0, 0, c) \geq V^*(m^*(p, 0, 0, c); p, 0, 0, c) .
\]

Since the cost is increasing in each component of the cash expenditures:

\[
V^*(m'; p, \kappa, z, c) \geq V^*(m'; p, 0, 0, c) \geq V^*(m^*(p, 0, 0, c); p, 0, 0, c) .
\]
Collecting these inequalities we have that for any $m'$:

\[
V^* (m^*(p, 0, 0, c); p, 0, 0, c) \leq V^* (m'; p, 0, 0, c) \leq V^* (m'; p, \kappa, z, c) \\
\leq V^* (m^*(p + \kappa, 0, 0); p + \kappa, 0, 0, c) + \frac{kb}{r}
\]

Finally since we show in Alvarez and Lippi (2009) that $V^* (m^*(\cdot, 0, 0, c); \cdot, 0, 0, z)$ is continuous and $\kappa b/r$ is continuous on $\kappa$, we have that for any $c$, $b/R$, and $p$, there exist a $\kappa^*$ such that for all $\kappa < \kappa^*$,

\[
\left| V^* (m^*(p + \kappa, 0, 0); p + \kappa, 0, 0, c) + \frac{kb}{r} - V^* (m^*(p, 0, 0, c); p, 0, 0, c) \right| < \epsilon(z).
\]

We note that the absolute value in the previous expression is independent of $z$. Hence for all $\kappa < \kappa^*$:

\[
\left| V^* (m^*(p + \kappa, 0, 0); p + \kappa, 0, 0, c) + \frac{kb}{r} - V^* (m^*(p, \kappa, c, z); p, \kappa, z, c) \right| < \epsilon(z).
\]

Finally, since $\epsilon(z)$ is increasing in $z$, but the upper and lower bounds on $V^* (m^*(p, \kappa, c, z); p, \kappa, z, c)$ are not, we have that the conclusion of the proposition holds also for $z' > z$.

**A.8 Invariant distribution of cash holdings $h$ and associated statistics**

We derive the stationary distribution of cash holdings. The pdf of the invariant distribution $h$ solves the ODE

\[
h (m) (p + \kappa) = h' (m) (c + \pi m)
\]

for $m^* \geq m \geq m^* - z$ and the DDE

\[
h (m) (p + \kappa) = h' (m) (c + \pi m) + \kappa h (m + z)
\]

for $0 \leq m \leq m^* - z$.

**Proof of the ODE-DDE for $h$.**

Take a discrete time version of the law motion with time period of length $\Delta$. For $m^* \geq m \geq m^* - z - (c + m^* \pi) \Delta$ we have:

\[
h (m, t + \Delta) = (1 - (p + \kappa) \Delta) h (m + (c + m \pi) \Delta, t)
\]

and for $0 \leq m < m^* - z - (c + m^* \pi) \Delta$ we have:

\[
h (m, t + \Delta) = (1 - (p + \kappa) \Delta) h (m + (c + m \pi) \Delta, t) + \kappa \Delta h (m + z + (c + m \pi) \Delta, t)
\]

This gives:

\[
h (m) (p + \kappa) = h' (m) (c + \pi m)
\]
In steady state,
\[ h(m) = (1 - (p + \kappa) \Delta) \ h(m + (c + m\pi) \Delta) \]
and
\[ h(m) = (1 - (p + \kappa) \Delta) \ h(m + (c + m\pi) \Delta + \kappa \Delta) \ h(m + z + (c + m\pi) \Delta) \]
or
\[ h(m) = (1 - (p + \kappa) \Delta) \ [h(m) + h'(m) \Delta (c + \pi m) + o(\Delta)] \]
and
\[ h(m) = (1 - (p + \kappa) \Delta) \ [h(m) + h'(m) \Delta (c + \pi m) + \kappa \Delta] \ h(m + z + (c + m\pi) \Delta) \]
or
\[ h(m) ((p + \kappa)) = (1 - (p + \kappa) \Delta) \ [h'(m) (c + \pi m) + o(\Delta) \Delta] \]
and
\[ h(m) (p + \kappa) = (1 - (p + \kappa) \Delta) \ h'(m) (c + \pi m) + \frac{o(\Delta)}{\Delta} + \kappa \ h(m + z + (c + m\pi) \Delta) \]
and taking \( \Delta \to 0 \):
\[ h(m) (p + \kappa) = h'(m) (c + \pi m) \]
for \( m^* \leq m \leq m - z \) and
\[ h(m) (p + \kappa) = h'(m) (c + \pi m) + \kappa \ h(m + z) \]
for \( 0 \leq m \leq m^* - z \). QED.

**Characterization of \( h \)**

As in the case of the value function, we can further characterize \( h \) by splitting its support \([0, m^*] \) into \( J \) intervals, where \( J \) is the smallest integer for which \( Jz \geq m^* \). The first \( J - 1 \) intervals have width \( z \), and are given by \([m^* - (j + 1) z, m^* - zj] \), for \( j = 0, 1, ..., J - 2 \), so that
\[ h_j : [m^* - (j + 1) z, m^* - zj] \to R_+ \]
for \( j = 0, ..., J - 2 \). The last one may be smaller, and is given by \([0, m^* - z (J - 1)] \), so that
\[ h_{J-1} : [0, m^* - z (J - 1)] \to R_+ \]
For the first interval we have an ODE
\[ h_0 (m) (p + \kappa) = h'_0 (m) (c + \pi m) \]
for \( m \in [m^* - z, m^*] \). Notice that, except for a multiplicative constant of integration, \( h_0 \) can be solved for in this interval. For the next intervals we take as given \( h_{J-1} \) and solve for \( h_j \) solving the following ode:
\[ h_j (m) (p + \kappa) = h'_j (m) (c + \pi m) + \kappa \ h_{j-1} (m + z) \]
for $m \in \left[ \max \left\{ m^* - z(j + 1), 0 \right\}, m^* - zj \right]$. 

We impose that the function $h$ is continuous everywhere, so that

$$h_j (m^* - zj) = h_{j-1} (m^* - zj)$$

for $j = 1, 2, \ldots, J - 1$. Notice that this implies that the derivatives of $h_j$ and $h_{j-1}$ agree at these points for $j \geq 2$.

Hence, by splitting the domain in this way we turn the DDE into the solution of several ODE's.

Finally, since $h$ is a density we have:

$$1 = \int_0^{m^*} h(m) \, dm = \sum_{j=0}^{J-1} \int_{m^* - z(j+1)}^{m^* - zj} h_j (m) \, dm + \int_0^{m^* - z(J-1)} h_{J-1} (m) \, dm.$$ 

**Solution of $h$ for the case of $\pi = 0$.**

The solution for $h$ is of the following form:

$$h(m) \left( \frac{p + \kappa}{c} \right) = h'(m)$$

for $m^* \geq m \geq m^* - z$ and otherwise

$$h(m) = \frac{c}{p + \kappa} h'(m) + \frac{\kappa}{p + \kappa} h(m + z)$$

We have that

$$h_0 (m) = K_{00} \exp (\mu (m + z - m^*))$$

since

$$h'_0 (m) = \mu K_{00} \exp (\mu (m + z - m^*))$$

Thus the ODE is satisfied setting

$$\mu = \frac{p + \kappa}{c}.$$ 

for any value of $K_{00}$.

For $j = 1, 2, \ldots, J - 1$ we have

$$h_j (m) = \frac{c}{p + \kappa} h'_j (m) + \frac{\kappa}{p + \kappa} h_{j-1} (m + z)$$

for $m \in \left[ \max \left\{ m^* - z(j + 1), 0 \right\}, m^* - zj \right]$.

We guess the solution of the form:

$$h_j (m) = \exp (\mu [m + z(j + 1) - m^*]) \sum_{i=0}^{j} K_{j,i} (m + z(j + 1) - m^*)^i$$
and thus

\[ h_j'(m) = \exp (\mu [m + z (j + 1) - m^*]) \sum_{i=0}^{j} K_{j,i} \left[ \mu (m + z (j + 1) - m^*)^i + i (m + z (j + 1) - m^*)^{i-1} \right] \]

Replacing our guess in the ode:

\[
\exp (\mu [m + z (j + 2) - m^*]) \sum_{i=0}^{j+1} K_{j+1,i} (m - z (j + 2) - m^*)^i
\]

\[
= \frac{c}{p + \kappa} \exp (\mu [m + z (j + 2) - m^*]) \sum_{i=0}^{j+1} K_{j+1,i} \left[ \mu (m + z (j + 2) - m^*)^i + i (m + z (j + 2) - m^*)^{i-1} \right]
\]

\[
+ \frac{\kappa}{p + \kappa} \exp (\mu [m + z (j + 2) - m^*]) \sum_{i=0}^{j} K_{j,i} (m - z (j + 2) - m^*)^i
\]

Simplifying:

\[
\sum_{i=0}^{j+1} K_{j+1,i} (m - z (j + 2) - m^*)^i
\]

\[
= \frac{c}{p + \kappa} \sum_{i=0}^{j+1} K_{j+1,i} \left[ \mu (m + z (j + 2) - m^*)^i + i (m + z (j + 2) - m^*)^{i-1} \right]
\]

\[
+ \frac{\kappa}{p + \kappa} \sum_{i=0}^{j} K_{j,i} (m - z (j + 2) - m^*)^i
\]

or using \( \mu = (p + \kappa) / c \):

\[
0 = \frac{c}{p + \kappa} \sum_{i=0}^{j+1} K_{j+1,i} i (m + z (j + 2) - m^*)^{i-1}
\]

\[
+ \frac{\kappa}{p + \kappa} \sum_{i=0}^{j} K_{j,i} (m - z (j + 2) - m^*)^i
\]

Matching the coefficients of term with \( (m - z (j + 1) - m^*)^{i-1} \) for \( i = 1, 2, ..., j + 1 \)

\[
\frac{c}{p + \kappa} K_{j+1,i} i = -\frac{\kappa}{p + \kappa} K_{j,i-1}
\]

or

\[
K_{j+1,i} = -\frac{1}{i c} K_{j,i-1}
\]

for \( i = 1, 2, ..., j + 1 \).
For $K_{j+1,0}$ we use that

$$h_{j+1}(m^* - z (j + 1)) = h_j(m^* - z (j + 1))$$

or

$$h_j(m^* - z (j + 1)) = \exp (\mu) \sum_{i=0}^{j} K_{j,i} 0^i = K_{j,0}$$

$$h_{j+1}(m^* - z (j + 1)) = \exp (\mu z) \sum_{i=0}^{j+1} K_{j+1,i} (z)^i$$

$$K_{j,0} = \exp (\mu z) \sum_{i=0}^{j+1} K_{j+1,i} (z)^i$$

or

$$K_{j+1,0} = \frac{K_{j,0}}{\exp (\mu z)} - \sum_{i=1}^{j+1} K_{j+1,i} (z)^i$$

Finally $K_{00}$ is obtained by requiring that

$$1 = \int_0^{m^*} h(m) \, dm = \sum_{j=0}^{J-1} \int_{m^* - z(j+1)}^{m^* - zj} h_j(m) \, dm + \int_0^{m^* - z(J-1)} h_{J-1}(m) \, dm.$$  

We use that for $0 \leq j < J - 1,$

$$\int_{m^* - z(j+1)}^{m^* - zj} h_j(m) \, dm$$

$$= \int_{m^* - z(j+1)}^{m^* - zj} \sum_{i=0}^{j} \left( \exp (\mu [m + z (j + 1) - m^*]) K_{j,i} (m + z (j + 1) - m^*)^i \right) dm$$

$$= \sum_{i=0}^{j} K_{j,i} \int_{m^* - z(j+1)}^{m^* - zj} \left( \exp (\mu [m + z (j + 1) - m^*]) (m + z (j + 1) - m^*)^i \right) dm$$

$$= \sum_{i=0}^{j} K_{j,i} \int_0^{\hat{m}} \exp (\mu \hat{m}) (\hat{m})^i \, d\hat{m}$$
and
\[
\int_0^{m^*-z(J-1)} h_{J-1}(m) \, dm
\]
\[
= \int_0^{m^*-z(J-1)} \left( \sum_{i=0}^{J-1} \exp(\mu [m + z (J - 2) - m^*]) K_{J-1,i} (m + z (J - 2) - m^*)^i \right) \, dm
\]
\[
= \sum_{i=0}^{J-1} K_{J-1,i} \int_0^{m^*-z(J-1)} \exp(\mu [m + z (J - 2) - m^*]) (m + z (J - 2) - m^*)^i \, dm
\]
\[
= \sum_{i=0}^{J-1} K_{J-1,i} \int_0^{m^*-z(J-1)} \exp(\mu [m + z (J - 2) - m^*]) (m + z (J - 2) - m^*)^i \, dm
\]
we have
\[
1 = \sum_{j=0}^{J-2} \sum_{i=0}^{j} K_{j,i} \int_0^{z} \exp(\mu \hat{m}) (\hat{m})^i \, d\hat{m}
\]
\[
+ \sum_{i=0}^{J-1} K_{J-1,i} \int_0^{m^*-z(J-1)} \exp(\mu [m + z (J - 2) - m^*]) (m + z (J - 2) - m^*)^i \, dm
\]
Letting
\[
L(i) = \int_0^{z} \exp(\mu m) \, m^i \, dm = \frac{\exp(\mu m) \, m^i}{\mu} \bigg|_0^{z} - \frac{i}{\mu} \int_0^{z} \exp(\mu m) \, m^{i-1} \, dm
\]
\[
= \frac{\exp(\mu z) \, z^i}{\mu} - \frac{i}{\mu} L(i-1)
\]
so that
\[
L(i) = \frac{\exp(\mu z) \, z^i}{\mu} - \frac{i}{\mu} L(i-1)
\]
for \(i = 1, 2, \ldots\), with
\[
L(0) = \int_0^{z} \exp(\mu m) \, dm = \frac{\exp(\mu z)}{\mu}.
\]
Likewise letting

\[ G(i) = \int_{0}^{m^*-z(J-1)} \exp(\mu [m + z (J - 2) - m^*]) (m + z (J - 2) - m^*)^i \, dm \]

\[ = \frac{\exp(\mu [m + z (J - 2) - m^*]) (m + z (J - 2) - m^*)^i}{\mu} \int_{0}^{m^*-z(J-1)} \frac{-i}{\mu} \int_{0}^{z} \exp(\mu [m + z (J - 2) - m^*]) (m + z (J - 2) - m^*)^{i-1} \, dm \]

\[ = \frac{\exp(\mu z) (z)^i}{\mu} - \frac{\exp(\mu [z (J - 2) - m^*]) (z (J - 2) - m^*)^i}{\mu} \]

\[ - \frac{i}{\mu} G(i - 1) \]

with

\[ G(0) = \int_{0}^{m^*-z(J-1)} \exp(\mu [m + z (J - 2) - m^*]) \, dm \]

\[ = \frac{\exp(\mu z)}{\mu} - \frac{\exp(\mu [z (J - 2) - m^*])}{\mu} \]

Now we can write

\[ 1 = \sum_{j=0}^{J-2} \sum_{i=0}^{j} K_{j,i} L(i) + \sum_{i=0}^{J-1} K_{J-1,i} G(i) \]

**Expected number of withdrawals** \( n \)

The expected number of withdrawals \( n \) is obtained by computing the reciprocal of the expected time between withdrawals. To do so we first compute the expected time until the next withdrawal, as a function of the current level of cash \( m \). We denote such a function as \( t(m) \). Below we show that this function solves the following DDE:

\[ t(m) (p + \kappa) = 1 + t'(m) (c + \pi m) + \kappa t(m - z) \]

for \( m^* \geq m > z \) and the ODE

\[ t(m) (p + \kappa) = 1 + t'(m) (c + \pi m) \]

for \( 0 \leq m < z \), with \( t(0) = 0 \). Since after any withdrawal real balances go to \( m^* \), the expected time between successive withdrawals is \( t(m^*) \) and by the fundamental theorem of renewal theory the average number of withdrawals is:

\[ n = \frac{1}{t(m^*)} \]

**Proof.** Consider a discrete time version with periods of length \( \Delta \) of the system. In this
case the time until adjustment solves
\[ t (m) = (1 - (p + \kappa) \Delta) \left[ \Delta + t (m - \Delta c - \pi m \Delta) \right] + \kappa \Delta t \left(m - \Delta c - \pi m \Delta - z \right) \]
for \( m > z \) and
\[ t (m) = (1 - (p + \kappa) \Delta) \left[ \Delta + t (m - \Delta c - \pi m \Delta) \right] \]
for \( m < z \), with boundary condition
\[ t (0) = 0. \]
This law of motion gives:
\[ t (m) (1 - \left[ 1 - (p + \kappa) \Delta \right]) = (1 - (p + \kappa) \Delta) \left[ \Delta + t' (m) (c + \pi m) \Delta + o(\Delta) + \kappa \Delta t \left(m - \Delta c - \pi m \Delta - z \right) \right] + \kappa \Delta t \left(m - \Delta c - \pi m \Delta - z \right) \]
or
\[ t (m) (p + \kappa) = (1 - (p + \kappa) \Delta) \left[ 1 + t' (m) (c + \pi m) + \frac{o(\Delta)}{\Delta} \right] + \kappa t \left(m - \Delta c - \pi m \Delta - z \right) \]
and taking \( \Delta \to 0 \):
\[ t (m) (p + \kappa) = 1 + t' (m) (c + \pi m) + \kappa t (m - z) \]
for \( m^* \geq m > z \) and
\[ t (m) (p + \kappa) = 1 + t' (m) (c + \pi m) \]
for \( 0 \leq m < z \).

**Characterization of the solution for \( t \)**
As in the case of the value function, we solve for \( t (\cdot) \) by dividing the domain in \( J \) intervals, where again \( J \) is the smallest integer for which \( Jz \geq m^* \). The first \( J - 1 \) intervals are of length \( z \), denoted them by \([zj, z(j+1)]\) for \( j = 0, 1, \ldots, J - 2 \). The last interval is \([z(J - 1), m^*]\). We then find \( t_0 : [0, z] \to R_+ \) that solves the ODE:
\[ t_0 (m) = \frac{1}{p + \kappa} + t'_0 (m) \left( \frac{c + \pi m}{p + \kappa} \right) \]
for \( m \in [0, z] \), and given \( t_{j-1} \) we solve for \( t_j : [zj, z(j+1)] \to R_+ \)
\[ t_j (m) = \frac{1}{p + \kappa} + t'_j (m) \left( \frac{c + \pi m}{p + \kappa} \right) + \frac{\kappa}{p + \kappa} t_{j-1} (m - z) \]
for \( m \in [zj, z(j+1)] \) for \( j = 0, 1, \ldots, J - 2 \). For \( j = J - 1 \), the function \( t_{J-1} : [z(J - 1), m^*] \to R_+ \) solves the same ode than for the other \( j \geq 1 \). Finally the continuity of \( t \) requires that
\[ t_{j+1} (z (j + 1)) = t_j (z (j + 1)) \]
for \( j = 0, 1, \ldots, J - 2 \). Hence, by splitting the domain in this way we turn the solution of a DDE into the solution of several ODE’s.
Solving $t(m)$ for $\pi = 0$.

In this case we have:

$$t_0(m) = \frac{1}{p+\kappa} + t'_0(m) \left( \frac{c}{p+\kappa} \right)$$

for $m \in [0, z]$, and given $t_{j-1}$ we solve for $t_j$

$$t_j(m) = \frac{1}{p+\kappa} + t'_j(m) \left( \frac{c}{p+\kappa} \right) + \frac{\kappa}{p+\kappa} t_{j-1}(m-z)$$

for $m \in [z, z(j+1)]$ for $j = 0, 1, \ldots, J - 2$, and for $j = J - 2$, then $m \in [z(J-1), m^*]$.  

I. Solution for $j = 0$. We guess a solution for $t_0$ of the form:

$$t_0(m) = C_0 + T_{0,0} \exp(\mu m)$$

so that

$$C_0 + T_{0,0} \exp(\mu m) = \frac{1}{p+\kappa} + T_{0,0}\mu \exp(\mu m) \left( \frac{c}{p+\kappa} \right)$$

and hence:

$$\mu = \frac{p+\kappa}{c},$$

$$C_0 = \frac{1}{p+\kappa}.$$  

We impose that $t(0) = 0$ obtaining

$$\frac{1}{p+\kappa} + T_{00} \exp \left( \frac{p+\kappa}{c} 0 \right) = 0$$

or

$$T_{00} = -\frac{1}{p+\kappa}.$$  

II. Solution for $j \geq 1$. For $m \in [zj, z(j+1)]$ and $1 \leq j \leq J - 2$ or $m \in [m(J-1), m^*]$ we guess

$$t_j(m) = C_j + \exp(\mu (m-zj)) \sum_{i=0}^{j} T_{j,i} (m-zj)^i$$

and hence:

$$t'_j(m) = \exp(\mu (m-zj)) \sum_{i=0}^{j} T_{j,i} \left[ \mu (m-zj)^i + i (m-zj)^{i-1} \right]$$

64
Substituting in the ode we have:

\[ C_{j+1} + \sum_{i=0}^{j+1} T_{j+1,i} \exp(\mu (m - z (j + 1))) (m - z (j + 1))^i \]

\[ = \frac{1}{p + \kappa} + \left( \frac{c}{p + \kappa} \right) \exp(\mu (m - z j)) \sum_{i=0}^{j+1} T_{j+1,i} \left[ \mu (m - z (j + 1))^i + i (m - z (j + 1))^{i-1} \right] \]

\[ + \frac{\kappa}{p + \kappa} \left[ C_j + \exp(\mu (m - z (j + 1))) \sum_{i=0}^{j} T_{j,i} (m - z (j + 1))^i \right] \]

Matching coefficients we have the following conditions. For the constant:

\[ C_{j+1} = \frac{1}{p + \kappa} + \frac{\kappa}{p + \kappa} C_j \]

For \( \exp(\mu (m - z (j + 1))) \)

\[ \sum_{i=0}^{j+1} T_{j+1,i} (m - z (j + 1))^i \]

\[ = \left( \frac{c}{p + \kappa} \right) \sum_{i=0}^{j+1} T_{j+1,i} \left[ \mu (m - z (j + 1))^i + i (m - z (j + 1))^{i-1} \right] \]

\[ + \frac{\kappa}{p + \kappa} \sum_{i=0}^{j} T_{j,i} (m - z (j + 1))^i \]

and using \( \mu = \frac{p + \kappa}{c} \),

\[ 0 = \left( \frac{c}{p + \kappa} \right) \sum_{i=0}^{j+1} T_{j+1,i} i (m - z (j + 1))^{i-1} \]

\[ + \frac{\kappa}{p + \kappa} \sum_{i=0}^{j} T_{j,i} (m - z (j + 1))^i \]

Matching the coefficients of \( (m - z (j + 1))^i \):

\[ \left( \frac{c}{p + \kappa} \right) T_{j+1,i+1} (i + 1) = -\frac{\kappa}{p + \kappa} T_{j,i} \]

or

\[ T_{j+1,i+1} = \frac{1}{(i + 1)} \left( \frac{\kappa}{c} \right) T_{j,i} \]

for \( i = 0, 1, \ldots, j \).
Finally we use the continuity of \(t\) at the edges of the intervals

\[t_{j+1}(z(j+1)) = t_j(z(j+1))\]

for all \(j = 0, 1, ..., J - 2\). This gives

\[t_j(z(j+1)) = C_j + \exp(\mu z) \sum_{i=0}^{j} T_{j,i} (z)^i\]

\[t_{j+1}(z(j+1)) = C_{j+1} + \exp(\mu 0) \sum_{i=0}^{j+1} T_{j+1,i} (0)^i = C_{j+1} + T_{j+1,0}\]

thus

\[C_j + \exp(\mu z) \sum_{i=0}^{j} T_{j,i} (z)^i = C_{j+1} + T_{j+1,0}\]

or

\[T_{j+1,0} = C_j + \exp(\mu z) \sum_{i=0}^{j} T_{j,i} (z)^i - C_{j+1}\]

Finally evaluating \(t_{J-1}(\cdot)\) at \(m^*\) gives the desired quantity.

**Average Withdrawals**

We characterize the average withdrawal size \(W\). To do so, notice that we can divide the withdrawals in three types: i) those that happens when \(m = 0\), ii) those that happens because a jump in consumption, i.e. the arrival of a consumption jump when \(m \leq z\) and iii) those that happens because the arrival of a free opportunity to withdraw. In average ther are \(n\) withdrawals per unit of time, out of which \(p\) are of type iii), \(\kappa \int_0^z h(m) \, dm\) of type ii) and hence \(n - p - \kappa \int_0^z h(m) \, dm\) are of type i. The size of the withdrawals is different in each case, so the average withdrawal is given by:

\[W = \left[ \frac{n - p - \kappa \int_0^z h(m) \, dm}{n} \right] m^* + \frac{p}{n} \int_0^{m^*} (m^* - m) h(m) \, dm \]

\[+ \left[ \frac{\kappa \int_0^z h(m) \, dm}{n} \right] \frac{\int_0^z (m^* + z - m) h(m) \, dm}{\int_0^z h(m) \, dm}\]

or

\[W = \left[ \frac{n - p - \kappa \int_0^z h(m) \, dm}{n} \right] m^* + \frac{p}{n} \int_0^{m^*} (m^* - m) h(m) \, dm \]

\[+ \left[ \frac{\kappa}{n} \right] \int_0^z (m^* + z - m) h(m) \, dm\]

### A.9 Supplemental Material
Table 5: Cash management statistics across payment technology - Italy

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Source SHIW. Entries are sample means. The unit of observation is the household whose head is not self-employed.